mathematics

MDPI

Article

Realization of Extremal Spectral Data by Pentadiagonal Matrices

Hubert Pickmann-Soto [1,*], Susana Arela-Pérez [1], Charlie Lozano [2] and Hans Nina [3]

[1] Departamento de Matemática, Facultad de Ciencias, Universidad de Tarapacá, Arica 1000000, Chile; sarelap@academicos.uta.cl
[2] Carrera de Matemática, Facultad de Ciencias Puras y Naturales, Universidad Mayor de San Andrés, La Paz 0201, Bolivia; clozano@fcpn.edu.bo
[3] Departamento de Matemáticas, Facultad de Ciencias Básicas, Universidad de Antofagasta, Antofagasta 1240000, Chile; hans.nina@uantof.cl
* Correspondence: hpickmanns@academicos.uta.cl

Abstract: In this paper, we address the extremal inverse eigenvalue problem for pentadiagonal matrices. We provide sufficient conditions for their existence and realizability through new constructions that consider spectral data of its leading principal submatrices. Finally, we present some examples generated from the algorithmic procedures derived from our results.

Keywords: inverse eigenvalue problem; symmetric pentadiagonal matrices; nonsymmetric pentadiagonal matrices; leading principal submatrices

MSC: 15A18; 15A42; 65F15; 65F18

1. Introduction

This paper is concerned with the inverse extremal eigenvalues problem for pentadiagonal matrices. An $n \times n$ matrix $\mathcal{A} = (\mathbf{a}_{ij})$ is called pentadiagonal if $\mathbf{a}_{ij} = 0$ for $|i-j| > 2$, where $n \geq 5$ is required. In particular, we consider real symmetric pentadiagonal matrices of the form

$$\mathcal{P} = \begin{pmatrix} a_1 & b_1 & c_1 & 0 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & c_2 & 0 & \dots & 0 \\ c_1 & b_2 & a_3 & \ddots & \ddots & \ddots & \vdots \\ 0 & c_2 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & b_{n-2} & c_{n-2} \\ \vdots & \vdots & \ddots & \ddots & b_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & \dots & 0 & c_{n-2} & b_{n-1} & a_n \end{pmatrix}, \tag{1}$$

with $c_i > 0$ and $b_i > 0$, and real nonsymmetric pentadiagonal matrices

$$\widetilde{\mathcal{P}} = \begin{pmatrix} a_1 & b_1 & d_1 & 0 & 0 & \dots & 0 \\ c_1 & a_2 & b_2 & d_2 & 0 & \dots & 0 \\ e_1 & c_2 & a_3 & \ddots & \ddots & \ddots & \vdots \\ 0 & e_2 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & b_{n-2} & d_{n-2} \\ \vdots & \vdots & \ddots & \ddots & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & \dots & 0 & e_{n-2} & c_{n-1} & a_n \end{pmatrix}. \tag{2}$$

The extremal inverse eigenvalues problem consists of determining necessary and/or sufficient conditions for the existence and realization of a structured matrix from extremal

check for updates

Citation: Pickmann-Soto, H.; Arela-Pérez, S.; Lozano, C.; Nina, H. Realization of Extremal Spectral Data by Pentadiagonal Matrices. *Mathematics* **2024**, *12*, 2198. https://doi.org/10.3390/math12142198

Academic Editor: Qing-Wen Wang

Received: 30 May 2024
Revised: 22 June 2024
Accepted: 4 July 2024
Published: 12 July 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

spectral data of its leading principal submatrices, which can be the minimal or maximal eigenvalues of the leading principal submatrices and some eigenvectors associated with them. In this regard, several papers discuss this problem for different structured matrices and some applications [1–8]. In most cases, the reconstructed matrices are symmetric, and as known, these matrices have real eigenvalues (see [9]). This problem differs from the classical inverse eigenvalues problem in which spectral information of the entire matrix is usually considered only (see [10,11]).

Pentadiagonal matrices often arise in numerical analysis. In particular, it is used to solve the inverse problem of a vibrating beam [12–14]. In the literature, there are different works on calculating the inverse matrix, the determinant, the characteristic polynomial, the eigenvalues, and the eigenvectors of a pentadiagonal matrix [15–19]. However, most consider special pentadiagonal matrices, such as Toeplitz and Companion, or with some null entries on their first or second diagonal [20–24]. Boley and Golub were the first to study the inverse eigenvalue problem for these matrices. In [25], they propose the construction of pentadiagonal symmetric matrices by extending the result obtained by Hochstadt [26] for tridiagonal symmetric matrices, using the eigenvalues of three principal submatrices and the block Lanczos algorithm. Ghanbari, in [27], also considers three spectra associated with blocks of a pentadiagonal matrix. The procedure used is similar to that given by Gladwell in [13] to construct tridiagonal matrices partitioned by blocks. Li presents another procedure in [28], where a pentadiagonal matrix is reconstructed from the eigenvalues of the matrix, but has as the leading principal submatrix a given pentadiagonal matrix. Wang gives a very different proposal in [29], where they use three autopairs for the construction of a symmetric pentadiagonal matrix. In the nonsymmetric case, there are very few advances, as we know, only for H-symmetric pentadiagonal matrices [30].

Throughout this paper, we denote by $\mathcal{A}_j;\ j = 1, 2, \ldots, n$, the leading principal submatrix of a matrix $\mathcal{A}$ of the order n. The eigenvalues of $\mathcal{A}_j$ are denoted and arranged in increasing order: $\lambda_1^{(j)} \leq \lambda_2^{(j)} \leq \ldots \leq \lambda_j^{(j)}$. $P_j(\lambda) = \det(\lambda I_j - \mathcal{A}_j)$ denotes the characteristic polynomial of the submatrix $\mathcal{A}_j$, where I_j is the identity matrix of order j. The minimal eigenvalue $\lambda_1^{(j)}$ and maximal eigenvalue $\lambda_j^{(j)}$ of $\mathcal{A}_j$ are called extremal eigenvalues of the leading principal submatrix $\mathcal{A}_j$.

We are interested in constructing pentadiagonal matrices of the forms (1) and (2). For this purpose, we propose the following problems:

Problem 1. *Given the set of* $2n-1$ *real numbers,*

$$\left\{\lambda_1^{(n)}, \ldots, \lambda_1^{(j)}, \ldots, \lambda_1^{(2)}, \lambda_1^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_j^{(j)}, \ldots, \lambda_n^{(n)}\right\},$$

and a nonzero vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, *construct a symmetric pentadiagonal matrix* $\mathcal{P}$ *of the form (1), such that* $\lambda_1^{(j)}$ *and* $\lambda_j^{(j)}$ *are the extremal eigenvalues of the leading principal submatrix* $\mathcal{P}_j$, $j = 1, 2, \ldots, n$ *of* $\mathcal{P}$, *and* $\left(\lambda_n^{(n)}, \mathbf{x}\right)$ *is an eigenpair of* $\mathcal{P}$.

Problem 2. *Given the set of* $3n-3$ *real numbers,*

$$\left\{\lambda_1^{(n)}, \ldots, \lambda_1^{(j)}, \ldots, \lambda_1^{(2)}, \lambda_1^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_j^{(j)}, \ldots, \lambda_n^{(n)}, \ell_1, \ell_2, \ldots, \ell_{n-2}\right\},$$

and the vectors $\mathbf{x} = (x_1, \ldots, x_n)^T$ *and* $\mathbf{y} = (y_1, \ldots, y_{n-1})^T$, *construct a nonsymmetric pentadiagonal matrix* $\widetilde{\mathcal{P}}$ *of the form (2) with* $a_{i+2} = \ell_i;\ i = 1, \ldots, n-2$, *such that* $\lambda_1^{(j)}$ *and* $\lambda_j^{(j)}$ *are the extremal eigenvalues of the leading principal submatrix* $\widetilde{\mathcal{P}}_j$, $\left(\lambda_n^{(n)}, \mathbf{x}\right)$ *is an eigenpair of the matrix* $\widetilde{\mathcal{P}}$ *and* $\left(\lambda_{n-1}^{(n-1)}, \mathbf{y}\right)$ *is an eigenpair of the leading principal submatrix* $\widetilde{\mathcal{P}}_{n-1}$.

The following lemmas are fundamental in the proofs of the main results of this work.

Lemma 1 ([6]). *Let $P(\lambda)$ be a monic polynomial of degree n with all real zeroes. If λ_1 and λ_n are, respectively, the smallest and largest zero of $P(\lambda)$, then*

(1) If $\mu < \lambda_1$, then $(-1)^n P(\mu) > 0$.
(2) If $\mu > \lambda_n$, then $P(\mu) > 0$.

Lemma 2 (Theorem 10.1.1., [31]). *(Cauchy's interlacing theorem) Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of an $n \times n$ real symmetric matrix A and $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_{n-1}$ be the eigenvalues of an $(n-1) \times (n-1)$ principal submatrix of A, then*

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \ldots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n.$$

An immediate consequence of this lemma is that, for any symmetric matrix, particularly for a pentadiagonal matrix,

$$\lambda_1^{(n)} \leq \cdots \leq \lambda_1^{(j)} \leq \cdots \leq \lambda_1^{(2)} \leq \lambda_1^{(1)} \leq \lambda_2^{(2)} \leq \cdots \leq \lambda_j^{(j)} \leq \cdots \leq \lambda_n^{(n)}.$$

Lemma 3. *An $n \times n$ matrix nonsymmetric pentadiagonal $\widetilde{\mathcal{P}}$ of the form (2) is diagonally similar to a matrix symmetric pentadiagonal $\mathcal{P}$ of the form (1) if $b_i c_i > 0$, $d_i e_i > 0$ and*

$$\prod_{j=i}^{i+1} \frac{c_j}{b_j} = \frac{e_i}{d_i}, \quad i = 1, 2, \ldots, n-2. \tag{3}$$

Proof. The similarity is followed considering the diagonal matrix $D = diag\{\xi_1, \xi_2, ..., \xi_n\}$, where $\xi_i^2 = \dfrac{c_i c_{i+1} \cdots c_{n-1}}{b_i b_{i+1} \cdots b_{n-1}}$; $i = 1, 2, \ldots, n-1$ and $\xi_n = 1$. Then, $D\widetilde{\mathcal{P}}D^{-1}$ is a symmetric pentadiagonal matrix of the form (1). □

Note that the above result provides a procedure for generating nonsymmetric pentadiagonal matrices, whose eigenvalues of their leading principal submatrices are real, from a symmetric pentadiagonal matrix. Consequently, the matrices in question have the same extremal eigenvalues.

This paper is organized as follows: In Section 2, we discuss a solution to Problem 1 and provide sufficient conditions for the existence and construction of a pentadiagonal symmetric matrix. In Section 3, we study Problem 2 and obtain also sufficient conditions for the nonsymmetric case. Finally, in Section 4, we show some examples to illustrate the results.

2. Construction of Symmetric Pentadiagonal Matrices

The results in this section are related to the existence of a symmetric pentadiagonal matrix. We provide a new construction of such matrices from the extremal eigenvalues of the leading principal submatrices. We start with the following lemma that is fundamental to the development of our results.

Lemma 4. *Let $\mathcal{P}$ be an $n \times n$ symmetric pentadiagonal matrix of the form (1), and let $\mathcal{P}_j$ be the $j \times j$ leading principal submatrix of $\mathcal{P}$ with characteristic polynomial $P_j(\lambda) = \det(\lambda I_j - \mathcal{P}_j)$, $j = 1, 2, \ldots, n$. Then, the sequence $\{P_j(\lambda)\}_{j=1}^n$ satisfies the recurrence relation:*

$$P_1(\lambda) = \lambda - a_1, \tag{4}$$

$$P_2(\lambda) = (\lambda - a_2)P_1(\lambda) - b_1^2, \tag{5}$$

$$P_j(\lambda) = (\lambda - a_j)P_{j-1}(\lambda) - b_{j-1}^2 P_{j-2}(\lambda) - c_{j-2}^2 Q_{j-1}(\lambda) + 2b_{j-1}c_{j-2}R_{j-1}(\lambda), \quad j = 3, 4, \ldots, n, \tag{6}$$

where $Q_{j-1}(\lambda)$ and $R_{j-1}(\lambda)$ are the determinants of the submatrices resulting from eliminating the $(j-2)$-th row and column, and the $(j-1)$-th row and $(j-2)$-th column of submatrix $\lambda I_{j-1} - \mathcal{P}_{j-1}$, respectively.

Proof of Lemma 4. It is immediate by expanding $\det(\lambda I_j - \mathcal{P}_j)$. □

Before presenting our main results concerning Problem 1, we define the following notations:

$$\left.\begin{aligned}
\alpha_j &= P_{j-2}\left(\lambda_1^{(j)}\right)P_{j-1}\left(\lambda_j^{(j)}\right) - P_{j-2}\left(\lambda_j^{(j)}\right)P_{j-1}\left(\lambda_1^{(j)}\right)\\
\beta_j &= R_{j-1}\left(\lambda_j^{(j)}\right)P_{j-1}\left(\lambda_1^{(j)}\right) - R_{j-1}\left(\lambda_1^{(j)}\right)P_{j-1}\left(\lambda_j^{(j)}\right)\\
\gamma_j &= Q_{j-1}\left(\lambda_1^{(j)}\right)P_{j-1}\left(\lambda_j^{(j)}\right) - Q_{j-1}\left(\lambda_j^{(j)}\right)P_{j-1}\left(\lambda_1^{(j)}\right)\\
\delta_j &= \left(\lambda_j^{(j)} - \lambda_1^{(j)}\right)P_{j-1}\left(\lambda_1^{(j)}\right)P_{j-1}\left(\lambda_j^{(j)}\right)
\end{aligned}\right\} \tag{7}$$

for $j = 3, 4, \ldots, n$.

Theorem 1. *Let $2n-1$ be real numbers $\left\{\lambda_1^{(j)}, \lambda_j^{(j)}\right\}_{j=1}^{n}$ satisfying*

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(j)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_j^{(j)} < \cdots < \lambda_n^{(n)}. \tag{8}$$

If

$$\beta_j^2 - \alpha_j\gamma_j \neq 0 \tag{9}$$

for $j = 3, 4, \ldots, n$, where α_j, β_j, and γ_j are as in (7), there exists a symmetric pentadiagonal matrix $\mathcal{P}$ of the form (1), such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the extremal eigenvalues of the leading principal submatrix $\mathcal{P}_j$ of $\mathcal{P}$, $j = 1, 2, \ldots, n$.

Proof of Theorem 1. Suppose that $\left\{\lambda_1^{(j)}, \lambda_j^{(j)}\right\}_{j=2}^{n}$ satisfies (8). Determining the existence of a symmetric pentadiagonal matrix A with the required properties is equivalent to proving that the system of equations

$$\begin{aligned}
P_j(\lambda_1^{(j)}) &= 0\\
P_j(\lambda_j^{(j)}) &= 0, \ j = 1, 2, \ldots, n
\end{aligned} \tag{10}$$

has real solutions a_j, $j = 1, 2, \ldots, n$, b_j, $j = 1, 2, \ldots, n-1$ and c_j, $j = 1, 2, \ldots, n-2$, where the characteristic polynomials $P_j(\lambda)$, $j = 1, 2, \ldots, n$ satisfy Lemma 4.

It is clear that, from System (10) for $j = 1, 2$, and Lemma 1, we can obtain the required the entries a_1, a_2 and b_1.

Now, from System (10), for $j = 3, 4, \ldots, n$, we have

$$\begin{cases}
P_j(\lambda_1^{(j)}) = (\lambda_1^{(j)} - a_j)P_{j-1}(\lambda_1^{(j)}) - b_{j-1}^2 P_{j-2}(\lambda_1^{(j)}) - c_{j-2}^2 Q_{j-1}(\lambda_1^{(j)})\\
\qquad +2b_{j-1}c_{j-2}R_{j-1}(\lambda_1^{(j)}) = 0\\
P_j(\lambda_j^{(j)}) = (\lambda_j^{(j)} - a_j)P_{j-1}(\lambda_j^{(j)}) - b_{j-1}^2 P_{j-2}(\lambda_j^{(j)}) - c_{j-2}^2 Q_{j-1}(\lambda_j^{(j)})\\
\qquad +2b_{j-1}c_{j-2}R_{j-1}(\lambda_j^{(j)}) = 0.
\end{cases} \tag{11}$$

Solving (11), we obtain

$$\alpha_j X^2 + \gamma_j Y^2 + 2\beta_j XY + \delta_j = 0. \tag{12}$$

with $X = b_{j-1}$ and $Y = c_{j-2}$. Note that, by fixing Y, the discriminant of Equation (12) is

$$\triangle_X = 4[(\beta_j^2 - \alpha_j\gamma_j)Y^2 - \alpha_j\delta_j]. \tag{13}$$

Since $\alpha_j\delta_j < 0$, $\triangle_X > 0$ if

(i) $\beta_j^2 - \alpha_j\gamma_j > 0$, for all $Y \in \mathbb{R}$.

(ii) $\beta_j^2 - \alpha_j\gamma_j < 0$, for $Y \in \left(-\sqrt{\frac{\alpha_j\delta_j}{\beta_j^2 - \alpha_j\gamma_j}}, \sqrt{\frac{\alpha_j\delta_j}{\beta_j^2 - \alpha_j\gamma_j}}\right)$.

Thus, X exists in either case. Analogously, by fixing X, we obtain that Y also exists. Moreover, the point $(X, Y) = (b_{j-1}, c_{j-2})$ belongs to the conic

$$\mathcal{C} = \left\{(X, Y) \in \mathbb{R}^2 : \alpha_j X^2 + \gamma_j Y^2 + 2\beta_j XY + \delta_j = 0\right\},$$

which, by Lemma 1 and condition (9), is non-degenerate, non-empty and centered at the origin. Therefore, there exist positive numbers b_{j-1} and c_{j-2}, $j = 3, 4, \ldots, n$ satisfying (12).

Finally, from (11) and Lemma 1, the entries a_j, $j = 3, 4, \ldots, n$ are obtained. □

Theorem 2. *Let $2n-1$ be real numbers $\left\{\lambda_1^{(j)}, \lambda_j^{(j)}\right\}_{j=1}^n$ and the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, satisfying*

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(j)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_j^{(j)} < \cdots < \lambda_n^{(n)},$$

and

$$x_j x_{j+1} > 0, \;\; i = 1, 2, \ldots, n-1. \tag{14}$$

If

$$\beta_j^2 - \alpha_j\gamma_j \neq 0$$

for $j = 3, 4, \ldots n$, where α_j, β_j, γ_j and δ_j are as in (7), then there exists a unique symmetric pentadiagonal matrix $\mathcal{P}$ of the form (1), such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the extremal eigenvalues of the leading principal submatrix $\mathcal{P}_j$ of $\mathcal{P}$, $j = 1, 2, \ldots, n$, and $(\lambda_n^{(n)}, \mathbf{x})$ is the eigenpair of $\mathcal{P}$.

Proof of Theorem 2. From Theorem 1, by conditions (8) and (9), we have determined the existence of a pentadiagonal matrix with the required spectral properties.

Now, from System (10), for $j = 1, 2$, and Lemma 1, it is clear that

$$a_1 = \lambda_1^{(1)}, \tag{15}$$

$$b_1 = \sqrt{\left(\lambda_1^{(1)} - \lambda_1^{(2)}\right)\left(\lambda_2^{(2)} - \lambda_1^{(1)}\right)} \tag{16}$$

and

$$a_2 = \lambda_1^{(2)} + \lambda_2^{(2)} - \lambda_1^{(1)}. \tag{17}$$

Now, from the second equality in (10), it follows that

$$\left.\begin{aligned} a_1x_1 + b_1x_2 + c_1x_3 &= \lambda_n^{(n)}x_1 \\ b_1x_1 + a_2x_2 + b_2x_3 + c_2x_4 &= \lambda_n^{(n)}x_2 \\ c_{j-2}x_{j-2} + b_{j-1}x_{j-1} + a_jx_j + b_jx_{j+1} + c_jx_{j+2} &= \lambda_n^{(n)}x_j, \quad j = 3, 4, \ldots, n-2 \\ c_{n-3}x_{n-3} + b_{n-2}x_{n-2} + a_{n-1}x_{n-1} + b_{n-1}x_n &= \lambda_n^{(n)}x_{n-1} \\ c_{n-2}x_{n-2} + b_{n-1}x_{n-1} + a_nx_n &= \lambda_n^{(n)}x_n. \end{aligned}\right\} \tag{18}$$

Thus, from (18) and condition (14), we obtain that

$$c_{j-2} = (\lambda_n^{(n)} - a_{j-2})\frac{x_{j-2}}{x_j} - c_{j-4}\frac{x_{j-4}}{x_j} - b_{j-3}\frac{x_{j-3}}{x_j} - b_{j-2}\frac{x_{j-1}}{x_j} \tag{19}$$

for $j = 3, 4, \ldots, n$, where $c_{-1} = c_0 = x_{-1} = x_0 = b_0 = 0$.

On the other hand, from System (10), for $j = 3, 4, \ldots, n$, we have

$$\alpha_j b_{j-1}^2 + \gamma_j c_{j-2}^2 + 2\beta_j b_{j-1} c_{j-2} + \delta_j = 0.$$

Then, from (19), Condition (9) and Lemma 1, we have

$$b_{j-1} = \frac{-c_{j-2}\beta_j \pm \sqrt{c_{j-2}^2\beta_j^2 - \alpha_j(c_{j-2}^2\gamma_j + \delta_j)}}{\alpha_j} \tag{20}$$

and

$$a_j = \lambda_i^{(j)} - \frac{b_{j-1}^2 P_{j-2}(\lambda_i^{(j)}) + c_{j-2}^2 Q_{j-1}(\lambda_i^{(j)}) - 2b_{j-1}c_{j-2}R_{j-1}(\lambda_i^{(j)})}{P_{j-1}(\lambda_i^{(j)})} \tag{21}$$

for $j = 3, 4, \ldots, n$, and $i = 1 \vee j$. This concludes the proof. □

3. Construction of Nonsymmetric Pentadiagonal Matrix

In this section, we give sufficient conditions for the existence of a nonsymmetric full pentadiagonal matrix. It should be noted that the construction given in Theorem 3 and the similarity process of Lemma 3 provide a procedure to construct a nonsymmetric full pentadiagonal matrix. However, this has the disadvantage that only uniqueness is obtained in the diagonal entries, in contrast to the procedure presented below. An extension of Lemma 4 to the nonsymmetric case is given in the following lemma.

Lemma 5. *Let $\widetilde{\mathcal{P}}$ be an $n \times n$ nonsymmetric pentadiagonal matrix of the form (2), and let $\widetilde{\mathcal{P}}_j$ be the $j \times j$ principal submatrix of $\widetilde{\mathcal{P}}$ with characteristic polynomial $P_j(\lambda) = \det(\lambda I_j - \widetilde{\mathcal{P}}_j)$, $j = 1, 2, \ldots, n$. Then, the sequence $\{P_j(\lambda)\}_{j=1}^n$ satisfies the recurrence relation:*

$$\begin{aligned} P_1(\lambda) &= \lambda - a_1, && (22)\\ P_2(\lambda) &= (\lambda - a_2)P_1(\lambda) - b_1c_1, && (23)\\ P_j(\lambda) &= (\lambda - a_j)P_{j-1}(\lambda) - b_{j-1}c_{j-1}P_{j-2}(\lambda) - d_{j-2}e_{j-2}Q_{j-1}(\lambda) \\ &\quad + b_{j-1}e_{j-2}R_{j-1}(\lambda) + c_{j-1}d_{j-2}T_{j-1}(\lambda);\ j = 3, 4, \ldots, n, && (24) \end{aligned}$$

where $Q_{j-1}(\lambda)$, $R_{j-1}(\lambda)$, and $T_{j-1}(\lambda)$ are the determinants of the submatrices, resulting from eliminating the $(j-2)$-th row and column, the $(j-1)$-th row and $(j-2)$-th column, and the $(j-2)$-th row and $(j-1)$-th column of submatrix $\lambda I_{j-1} - \widetilde{\mathcal{P}}_{j-1}$, respectively.

Proof of Lemma 5. It is immediate by expanding $\det(\lambda I_j - \widetilde{\mathcal{P}}_j)$. □

In the following, we consider the notations

$$\left.\begin{aligned} \eta_{j-1} &= b_{j-1}P_{j-2}\left(\lambda_1^{(j)}\right) - d_{j-2}T_{j-1}\left(\lambda_1^{(j)}\right) \\ \theta_{j-1} &= d_{j-2}Q_{j-1}\left(\lambda_1^{(j)}\right) - b_{j-1}R_{j-1}\left(\lambda_1^{(j)}\right) \\ \mu_{j-1} &= b_{j-1}P_{j-2}\left(\lambda_j^{(j)}\right) - d_{j-2}T_{j-1}\left(\lambda_j^{(j)}\right) \\ \tau_{j-1} &= d_{j-2}Q_{j-1}\left(\lambda_j^{(j)}\right) - b_{j-1}R_{j-1}\left(\lambda_j^{(j)}\right) \\ &\text{for } j = 3, 4, \ldots, n, \end{aligned}\right\} \tag{25}$$

and

$$z_{h,k} = x_h y_k - x_k y_h \quad \text{for } h = j+1, j+2 \text{ and } k = j-2, j-1, j. \tag{26}$$

Theorem 3. *Let $2n-1$ be real numbers $\left\{\lambda_1^{(j)}, \lambda_j^{(j)}\right\}_{j=1}^{n}$, the vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_{n-1})$, and real nonnegative numbers $\ell_1, \ell_2, \ldots, \ell_{n-2}$, satisfying*

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(j)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_j^{(j)} < \cdots < \lambda_n^{(n)},$$

$$\begin{aligned} x_j x_{j+1} &> 0, \quad j = 1, 2, \ldots, n-1, \\ y_j y_{j+1} &> 0, \quad j = 1, 2, \ldots, n-2 \end{aligned} \tag{27}$$

and

$$\frac{x_{j+2}}{x_{j+1}} \neq \frac{y_{j+2}}{y_{j+1}}, \; j = 1, 2, \ldots, n-3. \tag{28}$$

If

$$\mu_{j-1}\theta_{j-1} - \tau_{j-1}\eta_{j-1} \neq 0 \tag{29}$$

for $j = 3, 4, \ldots n$, where η_j, θ_j, μ_j and τ_j are as in (25), then there exists a nonsymmetric pentadiagonal matrix $\widetilde{\mathcal{P}}$ of the form (2), such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the extremal eigenvalues of the leading principal submatrix $\widetilde{\mathcal{P}}_j$ of $\widetilde{\mathcal{P}}$, $j = 1, 2, \ldots, n$, $(\lambda_n^{(n)}, \mathbf{x})$ is the eigenpair of $\widetilde{\mathcal{P}}$, $(\lambda_{n-1}^{(n-1)}, \mathbf{y})$ is the eigenpair of $\widetilde{\mathcal{P}}_{n-1}$, and $a_{i+2} = \ell_i$; $i = 1, \ldots, n-2$.

Proof of Theorem 3. Suppose that the set $\left\{\lambda_1^{(j)}, \lambda_j^{(j)}\right\}_{j=1}^{n}$, and the vectors $\mathbf{x}$ and $\mathbf{y}$, satisfy (8) and (28), respectively. To show the existence of a symmetric pentadiagonal matrix $\widetilde{\mathcal{P}}$ with the required properties is equivalent to showing that the system of equations

$$\begin{cases} P_j\left(\lambda_i^{(j)}\right) = 0, & \text{for } j = 1, 2, \ldots, n, \text{ and } i = 1, j, \\ \widetilde{\mathcal{P}}\mathbf{x} = \lambda_n^{(n)}\mathbf{x}, & \\ \widetilde{\mathcal{P}}\mathbf{y} = \lambda_{n-1}^{(n-1)}\mathbf{y}, & \end{cases} \tag{30}$$

where $P_j(\lambda) = \det(\lambda I_j - \widetilde{\mathcal{P}}_j)$, $j = 1, 2, \ldots, n$ satisfies Lemma 5, has real solutions a_j, b_j, c_j, d_j and e_j.

From the System (30) for $j = 1$ and Condition (8), its is clear that $a_1 = \lambda_1^{(1)}$. Note that the second and third equalities in (30) have the form:

$$\left.\begin{aligned} a_1 x_1 + b_1 x_2 + d_1 x_3 &= \lambda_n^{(n)} x_1 \\ c_1 x_1 + a_2 x_2 + b_2 x_3 + d_2 x_4 &= \lambda_n^{(n)} x_2 \\ e_{j-2}x_{j-2} + c_{j-1}x_{j-1} + a_j x_j + b_j x_{j+1} + d_j x_{j+2} &= \lambda_n^{(n)} x_j, \qquad j = 3, 4, \ldots, n-2 \\ e_{n-3}x_{n-3} + c_{n-2}x_{n-2} + a_{n-1}x_{n-1} + b_{n-1}x_n &= \lambda_n^{(n)} x_{n-1} \\ e_{n-2}x_{n-2} + c_{n-1}x_{n-1} + a_n x_n &= \lambda_n^{(n)} x_n \\ a_1 y_1 + b_1 y_2 + d_1 y_3 &= \lambda_{n-1}^{(n-1)} y_1 \\ c_1 y_1 + a_2 y_2 + b_2 y_3 + d_2 y_4 &= \lambda_{n-1}^{(n-1)} y_2 \\ e_{j-2}y_{j-2} + c_{j-1}y_{j-2} + a_j y_j + b_j y_{j+1} + d_j y_{j+2} &= \lambda_{n-1}^{(n-1)} y_j, \qquad j = 3, 4, \ldots, n-3 \\ e_{n-4}y_{n-4} + c_{n-3}y_{n-3} + a_{n-2}y_{n-2} + b_{n-2}y_{n-1} &= \lambda_{n-1}^{(n-1)} y_{n-2} \\ e_{n-3}y_{n-3} + c_{n-2}y_{n-2} + a_{n-1}y_{n-1} &= \lambda_{n-1}^{(n-1)} y_{n-1}. \end{aligned}\right\} \tag{31}$$

Then, from (31) and Condition (28),

$$b_1 = \frac{\lambda_n^{(n)} x_1 y_2 - \lambda_{n-1}^{(n-1)} x_2 y_1 + a_1(x_2 y_1 - x_1 y_2)}{x_3 y_2 - x_2 y_3} \tag{32}$$

and

$$d_1 = \frac{\lambda_n^{(n)} x_1 y_3 - \lambda_{n-1}^{(n-1)} x_3 y_1 + a_1(x_3 y_1 - x_1 y_3)}{x_2 y_3 - x_3 y_2}. \tag{33}$$

From System (30) for $j = 2$, Condition (8), and Lemma 1, we obtain

$$a_2 = \frac{\lambda_1^{(2)} P_1\left(\lambda_1^{(2)}\right) - \lambda_2^{(2)} P_1\left(\lambda_2^{(2)}\right)}{P_1\left(\lambda_1^{(2)}\right) - P_1\left(\lambda_2^{(2)}\right)} \tag{34}$$

and

$$c_1 = \frac{1}{b_1} \frac{\left(\lambda_2^{(2)} - \lambda_1^{(2)}\right) P_1\left(\lambda_1^{(2)}\right) P_1\left(\lambda_2^{(2)}\right)}{P_1\left(\lambda_1^{(2)}\right) - P_1\left(\lambda_2^{(2)}\right)}. \tag{35}$$

Now, from (31) and Condition (28), we have

$$b_j = \frac{\lambda_n^{(n)} x_j y_{j+2} - \lambda_{n-1}^{(n-1)} x_{j+2} y_j + a_j z_{j+2,j} + c_{j-1} z_{j+2,j-1} + e_{j-2} z_{j+2,j-2}}{x_{j+1} y_{j+2} - x_{j+2} - y_{j+1}} \tag{36}$$

and

$$d_j = \frac{\lambda_n^{(n)} x_j y_{j+1} - \lambda_{n-1}^{(n-1)} x_{j+1} y_j + a_j z_{j+1,j} + c_{j-1} z_{j+1,j-1} + e_{j-2} z_{j+1,j-2}}{x_{j+2} y_{j+1} - x_{j+1} - y_{j+2}} \tag{37}$$

for $j = 2, 3, \ldots, n-3$, where $e_0 = x_0 = y_0 = 0$.

On the other hand, from System (30) for $j = 3, 4, \ldots, n$, and Condition (29), it follows that

$$c_{j-1} = \frac{(\lambda_j^{(j)} - a_j) P_{j-1}(\lambda_j^{(j)}) \theta_{j-1} - (\lambda_1^{(j)} - a_j) P_{j-1}(\lambda_1^{(j)}) \tau_{j-1}}{\mu_{j-1} \theta_{j-1} - \tau_{j-1} \eta_{j-1}} \tag{38}$$

and

$$e_{j-2} = \frac{(\lambda_j^{(j)} - a_j) P_{j-1}(\lambda_j^{(j)}) \eta_{j-1} - (\lambda_1^{(j)} - a_j) P_{j-1}(\lambda_1^{(j)}) \mu_{j-1}}{\tau_{j-1} \eta_{j-1} - \mu_{j-1} \theta_{j-1}}. \tag{39}$$

Finally, from (31), and Conditions (14) and (27), we obtain

$$b_{n-2} = (\lambda_{n-1}^{(n-1)} - a_{n-2}) \frac{y_{n-2}}{y_{n-1}} - c_{n-3} \frac{y_{n-3}}{y_{n-1}} - e_{n-4} \frac{y_{n-4}}{y_{n-1}}, \tag{40}$$

$$d_{n-2} = (\lambda_n^{(n)} - a_{n-2}) \frac{x_{n-2}}{x_n} - b_{n-2} \frac{x_{n-1}}{x_n} - c_{n-3} \frac{x_{n-3}}{x_n} - e_{n-4} \frac{x_{n-4}}{x_n} \tag{41}$$

and

$$b_{n-1} = (\lambda_n^{(n)} - a_{n-1}) \frac{x_{n-1}}{x_n} - c_{n-2} \frac{x_{n-2}}{x_n} - e_{n-3} \frac{x_{n-3}}{x_n}. \tag{42}$$

This completes the proof. □

4. Numerical Examples

The proofs of Theorems 2 and 3 provide algorithmic procedures for constructing symmetric and nonsymmetric pentadiagonal matrices from the information given in the respective statements. The Algorithms were implemented in Matlab R2023b. In this section, we present three examples showing the construction of these types of matrices satisfying the stated sufficiency conditions. To show the approximation of the spectral data obtained with respect to those initially, we consider the relative error

$$e_{\lambda} = \log\left(\frac{\left\|\boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}\right\|_2}{\|\boldsymbol{\lambda}\|_2}\right), e_{\mathbf{v}} = \log\left(\frac{\|\mathbf{v} - \widetilde{\mathbf{v}}\|_2}{\|\mathbf{v}\|_2}\right) \text{and } e_{\mathbf{A}} = \log\left(\frac{\left\|\mathbf{A} - \widetilde{\mathbf{A}}\right\|_2}{\|\mathbf{A}\|_2}\right),$$

where $\boldsymbol{\lambda}$ is the vector containing the information of the initially given eigenvalues and $\widetilde{\boldsymbol{\lambda}}$ of the eigenvalues of the constructed matrix; $\mathbf{v}$ represents the given eigenvector and $\widetilde{\mathbf{v}}$ the eigenvector associated with a maximal eigenvalue of some leading principal submatrix of the constructed matrix; and $\mathbf{A}$ represents the initially given matrix and $\widetilde{\mathbf{A}}$ the reconstructed matrix.

Example 1. *We use the spectral data given in Table 1, which satisfies the conditions of Theorem 2. Using the algorithmic procedure that follows from the proof of Theorem 2, we obtain the matrix*

$$\mathcal{P} = \begin{pmatrix} 1.5052 & 0.2277 & 0.9049 & & & & \\ 0.2277 & 2.3546 & 0.4357 & 0.9797 & & & \\ 0.9049 & 0.4357 & 1.1524 & 0.3111 & 0.4389 & & \\ & 0.97978 & 0.3111 & 4.2215 & 0.9234 & 0.1111 & \\ & & 0.4389 & 0.9234 & 0.9738 & 0.4302 & 0.2581 \\ & & & 0.1111 & 0.4302 & 1.1296 & 0.1848 \\ & & & & 0.2581 & 0.1848 & 0.8535 \end{pmatrix},$$

whose spectral data, compared with the given data, have the relative errors shown in Table 2.

Table 1. Spectral data of matrix $\mathcal{P}$.

j	1	2	3	4	5	6	7
$\lambda_1^{(j)}$	1.5062	1.4490	0.3876	0.3814	0.1965	0.1437	0.1265
$\lambda_j^{(j)}$	1.5062	2.4118	2.7674	4.7154	4.9216	4.9317	4.9328
x_j	−0.0661	−0.3679	−0.1576	−0.8828	−0.2302	−0.0527	−0.0169

Table 2. Relative errors in the construction of matrix $\mathcal{P}$.

e_{λ}	$e_{\mathbf{x}}$
2.0164×10^{-16}	1.5385×10^{-14}

Example 2. *It is know that, for the matrix $J = tridiag\{1, 2, 1\}$,*

$$J^2 = \begin{pmatrix} 5 & 4 & 1 & & & & \\ 4 & 6 & 4 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 6 & 4 & 1 \\ & & & 1 & 4 & 6 & 4 \\ & & & & 1 & 4 & 5 \end{pmatrix}.$$

We reconstruct J^2 from its extremal eigenvalues and eigenpair $(\lambda_n^{(n)}, \mathbf{x})$ for different values of n. The results are shown in Table 3.

Table 3. Relative errors in the reconstruction of matrix J^2.

n	e_λ	$e_{\tilde{x}}$	e_{J^2}
5	1.6018×10^{-16}	1.7324×10^{-15}	5.3559×10^{-15}
10	1.9755×10^{-16}	8.4638×10^{-14}	8.2951×10^{-13}
15	1.5504×10^{-16}	8.7831×10^{-12}	1.6018×10^{-10}
20	2.0819×10^{-16}	1.1785×10^{-08}	3.2366×10^{-07}
25	1.7147×10^{-16}	1.1212×10^{-04}	4.2308×10^{-03}

Example 3. *Considering the similarity process given in the proof of Lemma 3 and the condition (3), the matrix*

$$\widetilde{\mathcal{P}} = \begin{pmatrix} 8 & 4 & 64 & & & & \\ 1 & 9 & 20 & 32 & & & \\ 4 & 5 & 11 & 12 & 5 & & \\ & 2 & 3 & 12 & 2 & 3 & \\ & & 5 & 8 & 13 & 4 & 1 \\ & & & 48 & 16 & 15 & 3 \\ & & & & 64 & 48 & 16 \end{pmatrix}$$

is generated from matrix

$$\mathcal{P} = \begin{pmatrix} 8 & 2 & 16 & & & & \\ 2 & 9 & 10 & 8 & & & \\ 16 & 10 & 11 & 6 & 5 & & \\ & 8 & 6 & 12 & 4 & 12 & \\ & & 5 & 4 & 13 & 8 & 8 \\ & & & 12 & 8 & 15 & 12 \\ & & & & 8 & 12 & 16 \end{pmatrix}.$$

We reconstruct these matrices from the spectral data required by Theorems 2 and 3. Table 4 shows the given data, and Tables 5 and 6 show the results.

Table 4. Spectral data of matrices $\mathcal{P}$ and $\widetilde{\mathcal{P}}$.

j	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$\lambda_1^{(j)}$	8.0000	6.4384	−8.3931	−8.5420	−9.0139	−9.2627	−9.2636
$\lambda_j^{(j)}$	8.0000	10.5616	29.4510	32.4643	33.6106	36.4775	39.8823
x_j	0.1874	0.2344	0.3441	0.4280	0.4055	0.5309	0.4026
$\tilde{x}_j$	0.2110	0.1320	0.0969	0.0603	0.1142	0.2990	0.9070
$\tilde{y}_j$	−0.7099	−0.3984	−0.2910	−0.1381	−0.1851	−0.4464	

Table 5. Relative errors in the construction of matrix $\mathcal{P}$.

e_λ	$e_{\mathbf{x}}$	$e_{\mathcal{P}}$
3.2171×10^{-16}	6.0989×10^{-14}	2.1021×10^{-13}

Table 6. Relative errors in the reconstruction of matrix $\widetilde{\mathcal{P}}$.

e_λ	$e_{\tilde{\mathbf{x}}}$	$e_{\tilde{\mathbf{y}}}$	$e_{\widetilde{\mathcal{P}}}$
6.6467×10^{-16}	2.7823×10^{-14}	2.4069×10^{-14}	5.8654×10^{-13}

Example 4. *In the literature, the pentadiagonal symmetric Toeplitz matrices of the following form are well known:*

$$\mathcal{T} = \begin{pmatrix} a & b & c & 0 & 0 & \dots & 0 \\ b & a & b & c & 0 & \dots & 0 \\ c & b & a & \ddots & \ddots & \ddots & \vdots \\ 0 & c & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & b & c \\ \vdots & \vdots & \ddots & \ddots & b & a & b \\ 0 & 0 & \dots & 0 & c & b & a \end{pmatrix}. \tag{43}$$

In this example, we reconstruct these matrices where a, b, and c have arbitrary integer values between 1 and 20 and for different orders of the matrix. The results are shown in Table 7.

Table 7. Relative errors in the reconstruction of matrix $\mathcal{T}$.

n	a, b, c	e_λ	$e_{\mathbf{x}}$	$e_{\mathcal{T}}$
5	8, 3, 5	2.4139×10^{-16}	2.6120×10^{-15}	8.4554×10^{-15}
10	2, 1, 4	3.3670×10^{-16}	2.9225×10^{-15}	3.8833×10^{-14}
15	16, 7, 11	8.3783×10^{-16}	7.8274×10^{-15}	2.6996×10^{-13}
20	14, 14, 15	5.6996×10^{-16}	1.7073×10^{-13}	1.6396×10^{-11}
25	19, 4, 17	2.9743×10^{-15}	1.2533×10^{-13}	1.2488×10^{-11}
50	9, 3, 20	5.0462×10^{-15}	4.5855×10^{-11}	9.6061×10^{-9}

5. Conclusions

This paper provides new procedures to reconstruct symmetric and nonsymmetric pentadiagonal matrices of order n from the minimal and maximal eigenvalues of all their leading principal submatrices. In the symmetric case, it is also necessary to consider an eigenvector of the maximal eigenvalue of the matrix. In the nonsymmetric case, an eigenvector of the maximal eigenvalue of the leading principal submatrix of order $n - 1$ is additionally required. In both cases, we give sufficient conditions for the existence and realizability of such matrices. As our results are constructive, algorithmic procedures are generated to determine a solution matrix.

Author Contributions: Conceptualization, H.P.-S., S.A.-P., C.L. and H.N.; investigation, H.P.-S., S.A.-P., C.L. and H.N.; writing—original draft, S.A.-P. and C.L.; writing—review and editing, H.P.-S. and H.N. All authors have contributed equally to the work. All authors have read and agreed to the published version of the manuscript.

Funding: Hubert Pickmann-Soto was supported by Universidad de Tarapacá, Arica, Chile, Proyecto Mayor de Investigación Científica y Tecnológica UTA 4782-24. Susana Arela-Pérez was supported by Universidad de Tarapacá, Arica, Chile, Proyecto Mayor de Investigación Científica y Tecnológica UTA 4778-24. C. Lozano thanks for the support to the Project "Dinámicas de Control" of IIMAT of Universidad Mayor de San Andrés, La Paz, Bolivia. H. Nina and C. Lozano thanks for the support to the Programa Regional MATH-AMSUD 23-MATH-09 MORA DataS project. H. Nina also thanks the hospitality of IIMAT of Universidad Mayor de San Andrés, La Paz, Bolivia where part of this work was performed.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors would like to thank the referees for their valuable suggestions that improved the final version of this paper.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Arela-Pérez, S.; Nina, H.; Pantáz, J.; Pickmann-Soto, H.; Valero, E. Construction of Lefkovitch and doubly Lefkovitch matrices with maximal eigenvalues and some diagonal elements prescribed. *Linear Algebra Appl.* **2021**, *626*, 152–170. [CrossRef]
2. Arela-Pérez, S.; Egaña, J.; Pasten, G.; Pickmann-Soto, H. Extremal realization spectra by two acyclic matrices whose graphs are caterpillars. *Linear Multilinear Algebra* **2023**, *71*, 1657–1680. [CrossRef]
3. Arela-Pérez, S.; Lozano, C.; Nina, H.; Pickmann-Soto, H.; Rodriguez, J. The new inverse eigenvalue problems for periodic and generalized periodic Jacobi matrices from their extremal spectral data. *Linear Algebra Appl.* **2023**, *659*, 55–72. [CrossRef]
4. Li, Z.; Zhao, X. Inverse Eigenvalue Problem for Generalized Periodic Jacobi Matrices with Linear Relation. In Proceedings of the Third International Symposium on Intelligent Information Technology Application, Nanchang, China, 21–22 November 2009; pp. 18–20.
5. Liu, Z.; Wang, K.; Xu, C. Extremal inverse eigenvalue problem for symmetric doubly arrow matrices. *J. Appl. Math. Comput.* **2014**, *45*, 151–164. [CrossRef]
6. Pickmann, H.; Soto, R.L.; Ega na, J.; Salas, M. An inverse problem for symmetric tridiagonal matrices. *Comput. Math. Appl.* **2007**, *54*, 699–708. [CrossRef]
7. Pickmann-Soto, H.; Arela-Pérez, S.; Nina, H.; Valero, E. Inverse maximal eigenvalues problems for Leslie and doubly Leslie matrices. *Linear Algebra Appl.* **2020**, *592*, 93–112. [CrossRef]
8. Sharma, D.; Sen, M. Inverse eigenvalue problems for two special acyclic matrices. *Mathematics* **2016**, *4*, 12. [CrossRef]
9. Horn, H.; Johnson, C.R. *Matrix Analysis*; Cambridge University Press: Cambridge, UK, 1985.
10. Boley, D.; Golub, G.H. A survey of matrix inverse eigenvalue problems. *Inverse Probl.* **1987**, *3*, 595–622. [CrossRef]
11. Chu, M.T.; Golub, G.H. *Inverse Eigenvalue Problems: Theory, Algorithms, and Applications*; Oxford University Press: Oxford, UK, 2005.
12. Barcilon, V. On the multiplicity of solutions of the inverse problem for a vibrating beam. *SIAM J. Appl. Math* **1979**, *37*, 605–613. [CrossRef]
13. Gladwell, G.M.L. *Inverse Problems in Vibration, Kluwer Academic Publishers*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2004.
14. Moghaddam, M.R.; Mirzaei, H.; Ghanbari, K. On the generalized inverse eigenvalue problem of constructing symmetric pentadiagonal matrices from three mixed eigendata. *Linear Multilinear Algebra* **2015**, *63*, 1154–1166. [CrossRef]
15. Andelić, M.; da Fonseca, C.M. Some determinantal cosiderations for pentadiagonal matrices. *Linear Multilinear Algebra* **2021**, *69*, 3121–3129. [CrossRef]
16. Marr, R.B.; Vineyard, G.H. Five-diagonal Toeplitz determinants an their relation to Chebyshev polinomials. *SIAM J. Matrix Anal. Appl.* **1988**, *9*, 579–586. [CrossRef]
17. Elouafi, A.D.A.H.M. On the characteristic polynomial, eigenvectors and determinant of a pentadiagonal matrices. *Appl. Math. Comput.* **2008**, *198*, 634–642.
18. Kanal, M.E.; Baykara, N.A.; Demiralp, M. Theory and algorithm of the inversion method for pentadiagonal matrices. *J. Math. Chem.* **2012**, *50*, 289–299. [CrossRef]
19. Sweet, R.A. A Recursive Relation for the Determinant of a Pentadiagonal Matrix. *Commun. ACM* **1969**, *12*, 330–332. [CrossRef]
20. Alvarez, M.A.; Brondani, A.E.; Macedo França, F.A.; Medina C., L.A. Characteristic Polynomials and Eigenvalues for Certain Classes of Pentadiagonal Matrices. *Mathematics* **2020**, *8*, 1056. [CrossRef]
21. Arela-Pérez, S.; Lozano, C.; Nina, H.; Pickmann-Soto, H. Two Inverse Eigenproblems for Certain Symmetric and Nonsymmetric Pentadiagonal Matrices. *Mathematics* **2022**, *10*, 3054. [CrossRef]
22. Chu, M.T.; Diele, F.; Ragni, S. On the inverse problem of constructing symmetric pentadiagonal Toeplitz matrices from three largest eigenvalues. *Inverse Probl.* **2005**, *21*, 1879. [CrossRef]
23. Eastman, B.; Meulen, K.N.V. Pentadiagonal Companion Matrices. *Spec. Matrices* **2016**, *4*, 13–30. [CrossRef]
24. Fasino, D. Spectral and structural properties of some pentadiagonal symmetric matrices. *Calcolo* **1988**, *25*, 301–310. [CrossRef]
25. Boley, D.; Golub, G.H. Inverse eigenvalue problems for band matrices. In *Lecture Notes in Mathematics*; Springer: Berlin/Heidelberg, Germany, 1977; Volume 630, pp. 23–31.
26. Hochstadt, H. On the construction of a Jacobi matrix from spectral data. *Linear Algebra Appl.* **1974**, *8*, 435–446. [CrossRef]
27. Ghambari, K.; Mirzaei, H. Inverse eigenvalue problem for pentadiagonal matrices. *Inverse Probl. Sci. Eng.* **2014**, *22*, 530–542. [CrossRef]
28. Li, J.; Dong, L.; Li, G. A class of inverse eigenvalue problems for real symmetric banded matrices with odd bandwidth. *Linear Algebra Appl.* **2018**, *541*, 131–162. [CrossRef]
29. Wang, Z.; Dai, H. On the construction of a real symmetric five-diagonal matrix from its three eigenpairs. *Appl. Math. Comput.* **2006**, *175*, 597–608. [CrossRef]
30. Mirzaei, H.; Ghambari, K. Construction of H-Symmetric pentadiagonal matrices by three spectra. *Appl. Math. Sci. Eng.* **2022**, *30*, 61–74. [CrossRef]
31. Parlett, B.N. *The Symmetric Eigenvalue Problem*; Society for Industrial and Applied Mathematics: Philadelphia, PA, USA, 1998.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.