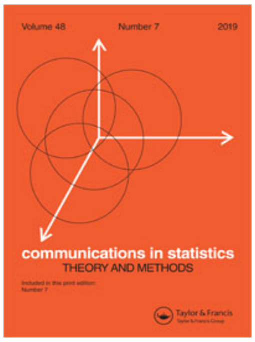

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Slashed power-Lindley distribution

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ABSTRACT
In this article, we introduce the slashed power-Lindley distribution. This model can be seen as an extension of the power-Lindley distribution with more flexibility in terms of the kurtosis of distribution. It arises as the ratio of two independent random variables, the one being a power-Lindley distribution and a power of the uniform distribution. We present properties and carry out estimates of the model parameters by the maximum likelihood method. Finally, we conduct a small simulation study to evaluate the performance of maximum likelihood estimators and we analyze a real data set to illustrate the usefulness of the new model.

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1. Introduction

A distribution that has attracted a wide application in survival and reliability is the Lindley (L) distribution. It was introduced by Lindley (1958) in the context of fiducial and Bayesian statistics. This distribution was also compounded with Poisson distribution, by Lindley (1965) and later by Sankaran (1970). In a recent study, Ghitany, Atieh, and Nadarajah (2008) have discussed Lindley distribution in detail with its applications and real data example.

A random variable X follows a Lindley distribution, denoted as $X \sim \mathrm{L}(\theta)$, if its probability density function (pdf) is given by

$$f_X(x;\theta) = \frac{\theta^2}{(\theta+1)}(1+x)e^{-\theta x}, \qquad x > 0 \text{ and } \theta > 0. \tag{1}$$

It is well known that the Lindley distribution is a two-component mixture of an exponential distribution (with scale θ) and a gamma distribution (with shape 2 and scale θ), with mixing proposition $p = \theta/(\theta+1)$.

At present, various extensions of the Lindley distribution can be found in the statistical literature. Ghittany et al. (2013) introduce a two-parameter extension of the Lindley distribution, the so-called power-Lindley distribution. This distribution provides more flexibility than the classical Lindley distribution in terms of the shape of the density as well as its skewness and kurtosis.

More specifically, a random variable X follows a power-Lindley distribution, denoted as $X \sim \mathrm{PL}(\theta,\alpha)$, if it can be represented as $X = Y^{1/\alpha}$, where $Y \sim \mathrm{L}(\theta)$ and $\alpha > 0$, and its pdf

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is given by

$$f_X(x;\theta,\alpha) = \frac{\alpha\theta^2}{\theta+1}(1+x^\alpha)x^{\alpha-1}e^{-\theta x^\alpha}, \qquad x>0, \tag{2}$$

where $\theta, \alpha > 0$ are shape parameters. Note that if $\alpha = 1$ the power-Lindley distribution reduces to the ordinary Lindley distribution. The r-th distributional moment of a power-Lindley random variable can be expressed as

$$\mu_r = E(X^r) = \frac{r\Gamma\left(\frac{r}{\alpha}\right)[\alpha(\theta+1)+r]}{\alpha^2\theta^{r/\alpha}(\theta+1)}, \qquad r=1,2,3,\ldots \tag{3}$$

Gui (2014) introduced a three-parameter extension called the Lindley-slash distribution. This extension is particularly more flexible than the Lindley distribution in terms of the kurtosis of distribution. In the context of slashed distributions with positive supports, a pioneering study is the one developed by Gómez, Olivares-Pacheco, and Bolfarine (2009), who extended the Birnbaum-Saunders distribution. Subsequently, the slash methodology has been used to extend other distributions, for example; the Weibull distribution (Olivares-Pacheco, Cornide-Reyes, and Monasterio 2010), half-normal distribution (Olmos et al. 2012), Rayleigh distribution (Iriarte et al. 2015), generalized exponential distribution (Astorga, Gómez, and Bolfarine 2017), among others.

A random variable T follows a Lindley-slash distribution, denoted as $T \sim \mathrm{LS}(\theta, q)$, if it can be represented as

$$T = \sigma\frac{X}{U^{1/q}}, \quad \sigma, q > 0, \tag{4}$$

where $X \sim \mathrm{L}(\theta)$ and $U \sim \mathrm{U}(0,1)$ are independent random variable, and its pdf is given by

$$f_T(t;\theta,q) = \frac{q\theta^2}{\sigma(\theta+1)}\int_0^1\left(1+\frac{ut}{\sigma}\right)e^{-\frac{\theta ut}{\sigma}}u^q\,du, \quad t>0, \tag{5}$$

where $\sigma > 0$ is a scale parameter, $\theta > 0$ is a shape parameter and $q > 0$ is a kurtosis parameter. Note that if $q \to \infty$ the Lindley-slash distribution tends to the Lindley distribution.

In this work, we propose an extension of the power-Lindley distribution with more flexibility in terms of the kurtosis of distribution, among other advantages. It arises as the quotient of two independent random variables, one being the power-Lindley distribution in numerator and a power of uniform distribution in the denominator.

The article is organized as follows. In Section 2 we present the stochastic representation, the density function and some mathematics properties of the new model. In Section 3, estimation of model parameters is discusses. In addition, it is conducted a simulation study to illustrate the behavior of the maximum likelihood estimates. In Section 4, an application to a real data set is presented. Final conclusions are reported.

2. Slash power-Lindley distribution

In this section, we study the main properties of the slashed power-Lindley distribution, such as: stochastic representation, density and hazard rate functions, ordinary moments and associated measures.

2.1. Stochastic representation

Definition 2.1. A random variable T follows a slashed power-Lindley distribution, denoted as $T \sim \mathrm{SPL}(\theta, \alpha, q)$, if it can be represented as

$$T = \frac{X}{U^{1/q}}, \quad q > 0, \tag{6}$$

where $X \sim \mathrm{PL}(\theta, \alpha)$ and $U \sim \mathrm{U}(0, 1)$ are independent random variable.

Remark 1.

i. If $q \to \infty$ then $T \to X$.
ii. If $q = 1$ then T follows a canonic slashed power-Lindley (CSPL) distribution.

2.2. Density function

Proposition 2.1. *Let $T \sim \mathrm{SPL}(\theta, \alpha, q)$, then the pdf of T can be written as*

$$f_T(t; \theta, \alpha, q) = \frac{qt^{-(q+1)}}{(\theta+1)\theta^{\frac{q}{\alpha}}}\left(\left(\frac{q}{\alpha}+\theta+1\right)\gamma\left(\frac{q}{\alpha}+1, \theta t^{\alpha}\right) - \theta^{\frac{q}{\alpha}+1}t^{q+\alpha}e^{-\theta t^{\alpha}}\right), \quad t > 0, \tag{7}$$

where $\theta > 0$ and $\alpha > 0$ are shape parameters, $q > 0$ is a kurtosis parameter and $\gamma(\alpha, x) = \int_0^x u^{\alpha-1}e^{-u}\,du$ is the incomplete gamma function.

Proof. By using the stochastic representation in (6) and the Jacobian method, we have the pdf of T, which can be expressed as

$$f_T(t; \theta, \alpha, q) = \frac{\alpha\theta^2}{(\theta+1)t}\int_0^1 (1 + (tw^{1/q})^{\alpha})(tw^{1/q})^{\alpha}e^{-\theta(tw^{1/q})^{\alpha}}\,dw\,.$$

Now, by letting $u = \theta(tw^{1/q})^{\alpha}$, the above pdf is reduces to

$$\begin{aligned} f_T(t; \theta, \alpha, q) &= \frac{q\theta\, t^{-(q+1)}}{(\theta+1)}\int_0^{\theta t^{\alpha}}\left(1+\frac{u}{\theta}\right)\left(\frac{u}{\theta}\right)^{\frac{q}{\alpha}} e^{-u}du \\ &= \frac{qt^{-(q+1)}}{\theta^{q/\alpha}(\theta+1)}\left(\theta\gamma\left(\frac{q}{\alpha}+1, \theta t^{\alpha}\right) + \gamma\left(\frac{q}{\alpha}+2, \theta t^{\alpha}\right)\right). \end{aligned}$$

Finally, the result is obtained by taking into account in the above equation that $\gamma(\alpha + 1, x) = \alpha\gamma(\alpha, x) - x^{\alpha}e^{-x}$ (Paris 2010). □

Remark 2. As $q = 1$ we obtain the canonic slashed power-Lindley distribution, which we denote as $T \sim \mathrm{CSPL}(\theta, \alpha)$. Hence, the density function of T is given by

$$\begin{aligned} &f_T(t; \theta, \alpha) \\ &= \frac{\alpha\, t^{-(\alpha+1)}}{(\theta+1)\theta^{\frac{1}{\alpha}}}\left(\left(\frac{1}{\alpha}+\theta+1\right)\gamma\left(\frac{1}{\alpha}+1, \theta\, t^{\alpha}\right) - \theta^{\frac{1}{\alpha}+1}t^{\alpha+1}e^{-\theta\, t^{\alpha}}\right), \quad t > 0, \quad \theta, \alpha > 0. \end{aligned}$$

Figure 1 depicts some of the shapes that the density function of the slashed power-Lindley distribution can take for different values of its parameters θ, α and q.

2.3. Some properties

In this subsection important properties of the slashed power-Lindley distribution are given.

Let $T \sim \mathrm{SPL}(\theta, \alpha, q)$, then

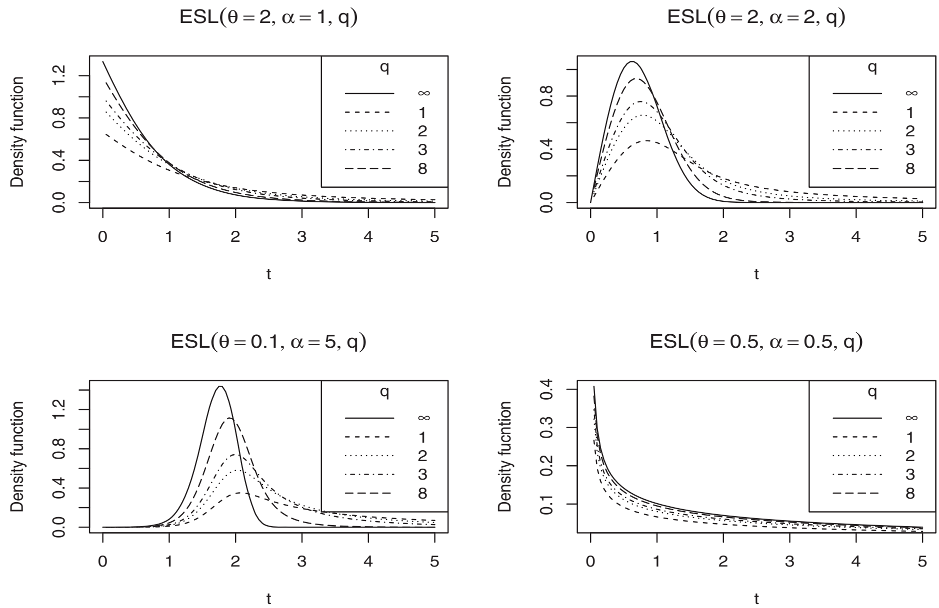

Figure 1. Plots of the density function of a SPL(θ, α, q) distribution for different values of its parameters.

(2.3.1) $\lim_{q\to\infty} f_T(t;\theta,\alpha,q) \;=\frac{\alpha\theta^2}{\theta+1}(1+t^\alpha)t^{\alpha-1}e^{-\theta t^\alpha}.$
(2.3.2) $\lim_{q\to\infty} f_T(t;\theta,1,q) \;=\frac{\theta^2}{\theta+1}(1+t)e^{-\theta t}.$
(2.3.3) $P(T\le t)=F_T(t;\theta,\alpha,q)=1-(1+\frac{\theta}{\theta+1}t^\alpha)e^{-\theta\, t^\alpha}-\frac{t}{q}f_T(t;\theta,\alpha,q).$

Remark 3. Property 2.3.1 reveal that as $q\to\infty$ the slashed power-Lindley distribution converges to the power-Lindley distribution (Ghittany et al. 2013).

Property 2.3.2 reveal that as $\alpha=1$ and $q\to\infty$ the slashed power-Lindley distribution converges to the classical Lindley distribution (Lindley 1958). Property 2.3.3 shows that the cumulative distribution function (cdf) of the slashed power-Lindley distribution can be written in terms of its probability density function.

Proposition 2.2. *Let* $T\sim \mathrm{SPL}(\theta,\alpha,q)$. *Then,*

(a) *The pdf of* $W=T^{-1}$ *is*

$$f_W(w;\theta,\alpha,q)$$
$$=\frac{qw^{q-1}}{(\theta+1)\theta^{q/\alpha}}\left(\left(\frac{q}{\alpha}+\theta+1\right)\gamma\left(\frac{q}{\alpha}+1,\frac{\theta}{w^\alpha}\right)-\frac{\theta^{\frac{q}{\alpha}+1}e^{-\frac{\theta}{w^\alpha}}}{w^{q+1}}\right),\quad w>0.$$

(b) *The pdf of* $W=\log(T)$ *is given by*

$$f_W(w;\theta,\alpha,q)$$
$$=\frac{qe^{-qw}}{(\theta+1)\theta^{q/\alpha}}\left(\left(\frac{q}{\alpha}+\theta+1\right)\gamma\left(\frac{q}{\alpha}+1,\theta w^{\alpha w}\right)-\theta^{\frac{q}{\alpha}+1}e^{(q+1)w}e^{-\theta e^{\alpha w}}\right),\quad w\in\mathbb{R}.$$

Proof. Parts (a)-(b) are directly obtained from the change-of-variable method. □

Remark 4. Part (a) demonstrates that these distributions are not closed under reciprocation, while the result in Part (b) can be used to study regression models in same lines as in the context of regression models for positive random variables; see McDonald and Butler(1990).

2.4. Reliability analysis

The reliability function $R_T(t)$, which is the probability of an item not failing prior to some time t, is defined by $R_T(t) = 1 - F_T(t)$. The reliability function of a slashed power-Lindley distribution is given by

$$R_T(t) = \left(1 + \frac{\theta}{\theta+1} t^{\alpha}\right) e^{-\theta\, t^{\alpha}} + \frac{t}{q} f_T(t; \theta, \alpha, q)\,,$$

where f_T is the pdf given in (7). A characteristic of interest of a random variable is its hazard rate function defined by $h_T(t) = \frac{f_T(t)}{1-F_T(t)}$ which is an important quantity characterizing the life-time of a certain phenomenon. It can be loosely interpreted as the conditional probability of failure at time t, given it has survived to time t. The hazard rate function for a slashed power-Lindley random variable is given by

$$h_T(t) = \frac{f_T(t; \theta, \alpha, q)}{\left(1 + \frac{\theta}{\theta+1} t^{\alpha}\right) e^{-\theta\, t^{\alpha}} + \frac{t}{q} f_T(t; \theta, \alpha, q)}\,,$$

where f_T is the pdf given in (7). Figure 2 displays some plots of the hazard rate function of a slashed power-Lindley distribution for different values of its parameters.

2.5. Moments and related measures

In this subsection, distributional moments of the slashed power-Lindley distribution are derived, an important need in any statistical analysis. Some of the important features and characteristics of a distribution can be studied through moments, which can be used to derive asymmetry and kurtosis coefficients.

Proposition 2.3. *Let $T \sim \mathrm{SPL}(\theta, \alpha, q)$. Then, for $r = 1, 2, \ldots$ and $q > r$ it follows that r-th moment is given by*

$$\mu_r = E(T^r) = \frac{r\Gamma(\frac{r}{\alpha})[\alpha(\theta+1)+r]}{\alpha^2(\theta+1)\theta^{r/\alpha}} \frac{q}{q-r}\,. \tag{8}$$

Proof. Using the stochastic representation given in (6), and since X and U are independent random variables, we have that

$$\mu_r = E(T^r) = E\left(\left(\frac{X}{U^{1/q}}\right)^r\right) = E\left(X^r\right) E\left(U^{-r/q}\right)\,,$$

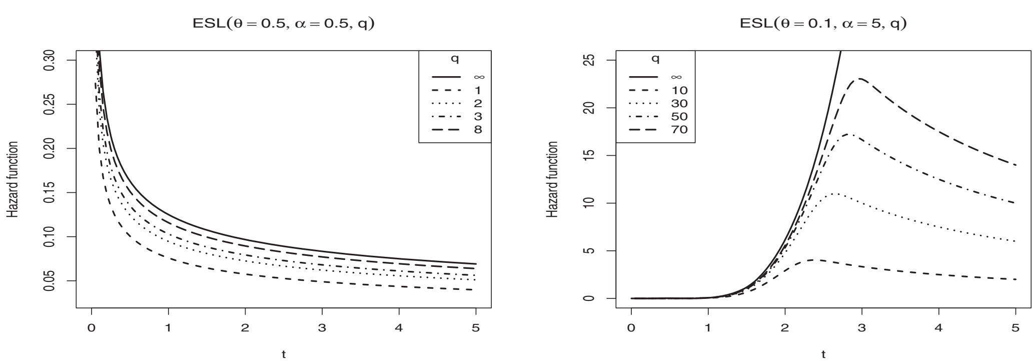

Figure 2. Plot of the hazard rate function of a slashed power-Lindley distribution.

where it follows that $E(U^{-r/q}) = \frac{q}{q-r}$, $q > r$ and $E(X^r) = \frac{r\Gamma(\frac{r}{\alpha})[\alpha(\theta+1)+r]}{\alpha^2(\theta+1)\theta^{r/\alpha}}$ are the moments for the PL(θ) distribution. □

Corollary 2.1. *Let $T \sim \text{SPL}(\theta, \alpha, q)$. Then, the mean and variance for $q > 1$ and $q > 2$ are given respectively by*

$$E(T) = \frac{\Gamma\left(\frac{1}{\alpha}\right)(A+2)}{\alpha(A+1)\theta^{1/\alpha}} \frac{q}{q-1} \quad \text{and}$$

$$Var(T) = \frac{q\left\{6\alpha(A+1)(q-1)\Gamma\left(\frac{2}{\alpha}\right) + A\left[2\alpha(A+1)(q-1)^2\Gamma\left(\frac{2}{\alpha}\right) - q(q-2)\Gamma^2\left(\frac{1}{\alpha}\right)A\right]\right\}}{(A+1)^2(q-1)^2(q-2)},$$

where $A = \alpha(\theta+1) - 1$.

Corollary 2.2. *Let $T \sim \text{SPL}(\theta, \alpha, q)$, then the asymmetry and kurtosis coefficients of T are given by*

$$\sqrt{\beta_1} = \frac{\mathbb{E}[T - \mathbb{E}(T)]^3}{[Var(T)]^{3/2}} = \frac{\mu_3 - 3\mu_2\mu_1 + 2\mu_1^3}{\left(\mu_2 - \mu_1^2\right)^{3/2}} \quad \text{and}$$

$$\beta_2 = \frac{\mathbb{E}[T - \mathbb{E}(T)]^4}{[Var(T)]^2} = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{\left(\mu_2 - \mu_1^2\right)^2},$$

respectively, where

$$\mu_1 = \frac{\Gamma\left(\frac{1}{\alpha}\right)[\alpha(\theta+1)+1]}{\alpha^2(\theta+1)\theta^{1/\alpha}} \frac{q}{q-1}, \quad q > 1,$$

$$\mu_2 = \frac{2\Gamma\left(\frac{2}{\alpha}\right)[\alpha(\theta+1)+2]}{\alpha^2(\theta+1)\theta^{2/\alpha}} \frac{q}{q-2}, \quad q > 2,$$

$$\mu_3 = \frac{3\Gamma\left(\frac{3}{\alpha}\right)[\alpha(\theta+1)+3]}{\alpha^2(\theta+1)\theta^{3/\alpha}} \frac{q}{q-3}, \quad q > 3,$$

$$\mu_4 = \frac{4\Gamma\left(\frac{4}{\alpha}\right)[\alpha(\theta+1)+4]}{\alpha^2(\theta+1)\theta^{4/\alpha}} \frac{q}{q-4}, \quad q > 4.$$

Figure 3 depict plots for the asymmetry and kurtosis coefficients of the slashed power-Lindley distribution. Here, it can be seen that the highest values of asymmetry and kurtosis are obtained for small values of the parameter q.

3. Inference

In this section we discuss the maximum likelihood estimation for parameters θ, α and q for the slashed power-Lindley distribution. Additionally, we conduct a small scale simulation study illustrating the ML estimations behavior for parameters θ, α and q in small and moderate sample sizes.

3.1. Maximum likelihood estimators

For a random sample $T_1, \ldots, T_n$ from the distribution SPL(θ, α, q), the log-likelihood function can be written as

$$\ell(\theta, \alpha, q) = n\log(q) + n\log(\theta) - n\log(\theta+1) - (q+1)\sum_{i=1}^{n}\log(t_i) + \sum_{i=1}^{n}\log M(t_i), \quad (9)$$

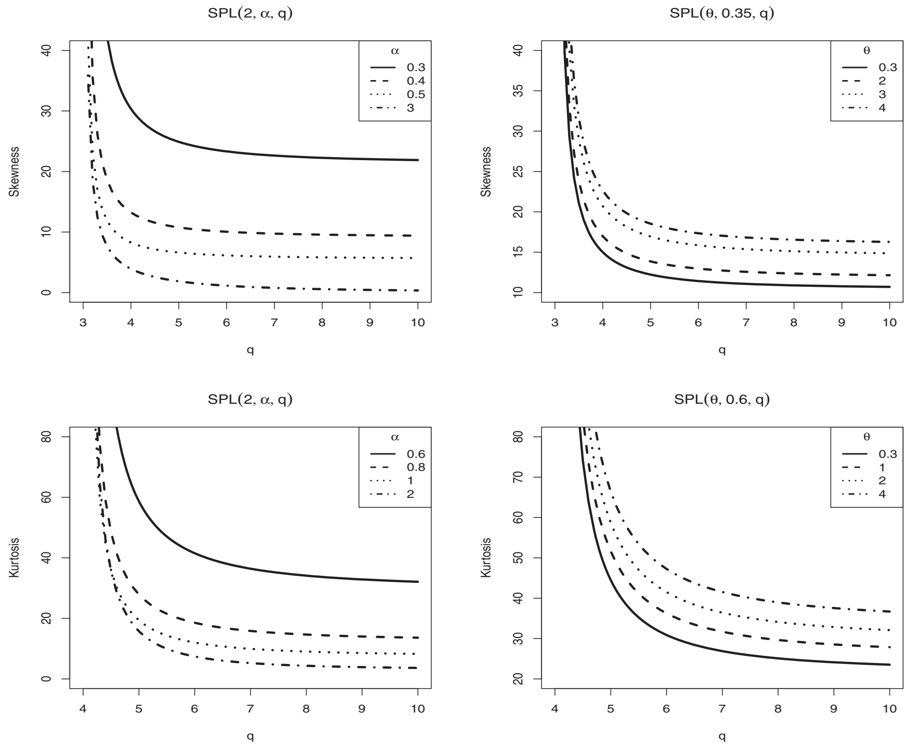

Figure 3. Plot of the asymmetry and kurtosis coefficients of a slashed power-Lindley distribution.

where

$$M(t_i) = \int_0^{\theta\, t_i^{\alpha}} \left(1 + \frac{u}{\theta}\right) \left(\frac{u}{\theta}\right)^{\frac{q}{\alpha}} e^{-u}\, du\,,$$

so that the maximum likelihood equations are given by

$$\frac{n}{\theta} - \frac{n}{\theta+1} + \sum_{i=1}^{n} \frac{M_1(t_i)}{M(t_i)} = 0,$$

$$\sum_{i=1}^{n} \frac{M_2(t_i)}{M(t_i)} = 0,$$

$$\frac{n}{q} - \sum_{i=1}^{n} \log(t_i) + \sum_{i=1}^{n} \frac{M_3(t_i)}{M(t_i)} = 0,$$

where

$$M_1(t_i) := \frac{\partial M(t_i)}{\partial \theta} = -\int_0^{\theta\, t_i^{\alpha}} \frac{q\theta + qu + \alpha\, u}{\alpha\theta^2} \left(\frac{u}{\theta}\right)^{\frac{q}{\alpha}} e^{-u}\, du + t_i^{q+\alpha} \left(1 + t_i^{\alpha}\right) e^{-\theta\, t_i^{\alpha}},$$

$$M_2(t_i) := \frac{\partial M(t_i)}{\partial \alpha} = -\int_0^{\theta\, t_i^{\alpha}} \frac{q}{\alpha^2} \left(\frac{u}{\theta}\right)^{\frac{q}{\alpha}} \log\left(\frac{u}{\theta}\right) \left(1 + \frac{u}{\theta}\right) e^{-u}\, du + \theta\, t_i^{q+\alpha} \left(1 + t_i^{\alpha}\right) \log(t_i) e^{-\theta\, t_i^{\alpha}},$$

$$M_3(t_i) := \frac{\partial M(t_i)}{\partial q} = \int_0^{\theta\, t_i^{\alpha}} \frac{1}{\alpha} \left(\frac{u}{\theta}\right)^{\frac{q}{\alpha}} \log\left(\frac{u}{\theta}\right) \left(1 + \frac{u}{\theta}\right) e^{-u}\, du.$$

It is well known that as the sample size increases, the distribution of the MLE tends (under regularity conditions) to the normal distribution with mean (θ, α, q) and covariance matrix equal to the inverse of the Fisher (expected) information matrix. Due to the complexity of the likelihood function it is not possible to obtain its analytical expression. It is possible, however, to work with the observed information matrix, which is a consistent estimator for the expected information matrix.

3.2. Observed information matrix

The observed information matrix follows from the Hessian matrix, replacing unknown parameters by their MLEs. Some algebraic manipulation yields the following Hessian matrix: Let $T \sim \text{SPL}(\theta, \alpha, q)$, so that the observed information matrix is given by

$$I_n(\theta,\alpha,q) = \begin{pmatrix} \frac{\partial^2 l(\theta,\alpha,q)}{\partial\theta^2} & \frac{\partial^2 l(\theta,\alpha,q)}{\partial\theta\partial\alpha} & \frac{\partial^2 l(\theta,\alpha,q)}{\partial\theta\partial q} \\ & \frac{\partial^2 l(\theta,\alpha,q)}{\partial\alpha^2} & \frac{\partial^2 l(\theta,\alpha,q)}{\partial\alpha\partial q} \\ & & \frac{\partial^2 l(\theta,\alpha,q)}{\partial^2 q} \end{pmatrix},$$

such that

$$\frac{\partial^2 l(\theta,\alpha,q)}{\partial\theta^2} = -\frac{n}{\theta^2} + \frac{n}{(\theta+1)^2} + \sum_{i=1}^{n}\frac{M_{11}(t_i)}{M(t_i)} - \sum_{i=1}^{n}\left(\frac{M_1(t_i)}{M(t_i)}\right)^2,$$

$$\frac{\partial^2 l(\theta,\alpha,q)}{\partial\theta\partial\alpha} = \sum_{i=1}^{n}\frac{M_{12}(t_i)}{M(t_i)} - \sum_{i=1}^{n}\frac{M_1(t_i)M_2(t_i)}{(M(t_i))^2},$$

$$\frac{\partial^2 l(\theta,\alpha,q)}{\partial\theta\partial q} = \sum_{i=1}^{n}\frac{M_{13}(t_i)}{M(t_i)} - \sum_{i=1}^{n}\frac{M_1(t_i)M_3(t_i)}{(M(t_i))^2},$$

$$\frac{\partial^2 l(\theta,\alpha,q)}{\partial\alpha^2} = \sum_{i=1}^{n}\frac{M_{22}(t_i)}{M(t_i)} - \sum_{i=1}^{n}\left(\frac{M_2(t_i)}{M(t_i)}\right)^2,$$

$$\frac{\partial^2 l(\theta,\alpha,q)}{\partial\alpha\partial q} = \sum_{i=1}^{n}\frac{M_{23}(t_i)}{M(t_i)} - \sum_{i=1}^{n}\frac{M_2(t_i)M_3(t_i)}{(M(t_i))^2},$$

$$\frac{\partial^2 l(\theta,\alpha,q)}{\partial q^2} = -\frac{n}{q} + \sum_{i=1}^{n}\frac{M_{33}(t_i)}{M(t_i)} - \sum_{i=1}^{n}\left(\frac{M_3(t_i)}{M(t_i)}\right)^2.$$

where

$$\begin{aligned} M_{11}(t_i) := \frac{\partial\, M_1(t_i)}{\partial\theta} &= \int_0^{\theta\, t_i^{\alpha}} \frac{(q+\alpha)(q\theta+qu+2\alpha\, u)}{\alpha^2\theta^3}\left(\frac{u}{\theta}\right)^{\frac{q}{\alpha}} e^{-u}\, du \\ &\quad - \frac{q+(q+\alpha)t_i^{\alpha}}{\alpha\theta} t_i^{q+\alpha} e^{-\theta\, t_i^{\alpha}} - t_i^{2\alpha+q}(1+t_i^{\alpha})e^{-\theta\, t_i^{\alpha}}, \\ M_{12}(t_i) := \frac{\partial\, M_1(t_i)}{\partial\theta} &= \int_0^{\theta\, t_i^{\alpha}} \frac{q}{\alpha^3\theta^2}\left(\frac{u}{\theta}\right)^{\frac{q}{\alpha}}\left(\log\left(\frac{u}{\theta}\right)(q\theta+qu+u\alpha)+\alpha(\theta+u)\right)e^{-u}\, du \\ &\quad - t_i^{\alpha+q}\log(t_i)e^{-\theta\, t_i^{\alpha}}\left(\frac{q+(q+\alpha)t_i^{\alpha}}{\alpha} - (1+t_i^{\alpha})\right) \\ &\quad + t_i^{2\alpha+q}\log(t_i)e^{-\theta\, t_i^{\alpha}}\left(1-\theta(1+t_i^{\alpha})\right), \\ M_{13}(t_i) := \frac{\partial\, M_1(t_i)}{\partial\theta} &= -\int_0^{\theta\, t_i^{\alpha}} \frac{1}{\alpha^2\theta^2}\left(\frac{u}{\theta}\right)^{\frac{q}{\alpha}}\left(\log\left(\frac{u}{\theta}\right)(q\theta+qu+\alpha\, u)+\alpha(\theta+u)\right) du \\ &\quad + t_i^{\alpha+q}\log(t_i)(1+t_i^{\alpha})e^{\theta\, t_i^{\alpha}}, \end{aligned}$$

$$M_{22}(t_i) := \frac{\partial M_2(t_i)}{\partial \theta} = \int_0^{\theta t_i^\alpha} \frac{q}{\alpha^4\theta}\left(\frac{u}{\theta}\right)^{\frac{q}{\alpha}} \log\left(\frac{u}{\theta}\right)(\theta+u)e^{-u}\left(q\log\left(\frac{u}{\theta}\right)+2\alpha\right)du$$
$$-\theta\, t_i^{\alpha+q}\log^2(\theta)(1+t_i^\alpha)\left(\frac{q}{\alpha}-1\right)e^{-\theta t_i^\alpha}$$
$$+\theta\, t_i^{2\alpha+q}\log^2(t_i)\left(1-\theta(1+t_i^\alpha)\right)e^{-\theta t_i^\alpha},$$
$$M_{23}(t_i) := \frac{\partial M_2(t_i)}{\partial \theta} = -\int_0^{\theta t_i^\alpha} \frac{1}{\alpha^3\theta}\left(\frac{u}{\theta}\right)^{\frac{q}{\alpha}} \log\left(\frac{u}{\theta}\right)(\theta+u)\left(q\log\left(\frac{u}{\theta}\right)+\alpha\right)e^{-u}\,du$$
$$+\theta\, t_i^{\alpha+q}\log^2(t_i)(1+t_i^\alpha)e^{-\theta t_i^\alpha},$$
$$M_{33}(t_i) := \frac{\partial M_3(t_i)}{\partial \theta} = \int_0^{\theta t_i^\alpha}\left(\frac{u}{\theta}\right)^{\frac{q}{\alpha}}\left(1+\frac{u}{\theta}\right)e^{-u}\,du.$$

3.3. Simulation study

In order to examine the performance of the proposed approach, we present a simulation study to assess the performance of the estimation procedure based on the ML approach for the parameters θ, α and q of the SPL model. The simulation study is conducted considering 1000 generated samples of size $n = 30, 50$ and 100 from the SPL model. The goal of this simulation is to study the behavior of the ML estimators for the model parameters using our proposed procedure. To generate $T \sim \mathrm{SPL}(\theta, \alpha, q)$ the following algorithm was used:

Step 1. Generate X having a $\mathrm{PL}(\theta, \alpha)$ distribution.

1. Generate $U \sim \mathrm{Uniform}(0, 1)$.
2. Compute $X = [-1 - \frac{1}{\theta} - \frac{1}{\theta}W_{-1}(-\frac{\beta+1}{e^{\beta+1}}(1-U))]^{1/\alpha}$.

where W_{-1} is the non-principal branch of the Lambert function, see (Ghitany et al. 2013; Corless et al. 1996).

Step 2. Generate T having a $\mathrm{SPL}(\theta, \alpha, q)$ distribution.

1. Generate $W \sim \mathrm{Uniform}(0, 1)$.
2. Compute $T = XW^{-1/q}$.

For each generated sample, the ML estimates were computed numerically for each parameter, and the mean value and standard deviation (SD) are computed. Table 1 shows the resulting ML estimates of the parameters (θ, α, q). It can be seen from Table 1 that the ML estimates are quite stable and close to the real values for the sample size considered. As expected, we also can see the mean square error decrease as sample size increases.

4. Application

In the following subsection, we illustrate an application to a real data set where it is shown that the model SPL fits the data studied better than models L, PL and LS.

4.1. Survival time data

The data set correspond to the survival times of guinea pigs (in days) injected with different doses of tubercle bacilli. This data set has been obtained from Bjerkedal (1960). Table 2 presents summary statistics for the survival times data set where b_1 y b_2 are sample asymmetry and kurtosis coefficients, respectively.

L, PL, LS and SPL distributions were fitted to the data set. As will be seen, the SPL distribution provided a better fit. Table 3 shows the ML estimates of the parameters, with the

Table 1. ML estimates for the parameters θ, α and q of the SPL model.

θ	α	q	$\widehat{\theta}$ (SD)	$\widehat{\alpha}$ (SD)	$\widehat{q}$ (SD)
			$n = 30$		
1.0	0.5	1.0	1.017 (0.176)	0.522 (0.094)	1.061 (0.198)
1.0	0.5	1.5	1.013 (0.150)	0.516 (0.075)	1.646 (0.409)
1.0	0.5	2.0	1.020 (0.152)	0.515 (0.064)	2.267 (1.076)
2.0	1.0	1.0	2.050 (0.458)	1.041 (0.190)	1.065 (0.188)
3.0	1.0	1.5	3.061 (0.639)	1.026 (0.119)	1.636 (0.386)
4.0	1.0	2.0	4.097 (0.789)	1.018 (0.092)	2.260 (0.647)
1.0	2.0	2.0	1.033 (0.216)	2.159 (0.599)	2.076 (0.331)
1.0	3.0	2.5	1.019 (0.234)	3.343 (1.090)	2.602 (0.407)
1.0	4.0	3.0	1.020 (0.231)	4.477 (1.557)	3.128 (0.478)
			$n = 50$		
θ	α	q	$\widehat{\theta}$ (SD)	$\widehat{\alpha}$ (SD)	$\widehat{q}$ (SD)
1.0	0.5	1.0	1.012 (0.131)	0.512 (0.066)	1.036 (0.154)
1.0	0.5	1.5	1.011 (0.120)	0.515 (0.055)	1.560 (0.274)
1.0	0.5	2.0	1.017 (0.115)	0.507 (0.048)	2.156 (0.493)
2.0	1.0	1.0	2.029 (0.360)	1.020 (0.142)	1.042 (0.146)
3.0	1.0	1.5	3.025 (0.491)	1.014 (0.093)	1.579 (0.261)
4.0	1.0	2.0	4.056 (0.605)	1.009 (0.071)	2.165 (0.475)
1.0	2.0	2.0	1.014 (0.162)	2.101 (0.412)	2.076 (0.257)
1.0	3.0	2.5	1.017 (0.175)	3.177 (0.667)	2.531 (0.296)
1.0	4.0	3.0	1.009 (0.176)	4.254 (0.972)	3.047 (0.366)
			$n = 100$		
θ	α	q	$\widehat{\theta}$ (SD)	$\widehat{\alpha}$ (SD)	$\widehat{q}$ (SD)
1.0	0.5	1.0	1.001 (0.092)	0.506 (0.045)	1.016 (0.107)
1.0	0.5	1.5	1.007 (0.087)	0.505 (0.037)	1.538 (0.200)
1.0	0.5	2.0	1.006 (0.084)	0.505 (0.036)	2.091 (0.333)
2.0	1.0	1.0	2.010 (0.244)	1.010 (0.097)	1.017 (0.099)
3.0	1.0	1.5	3.018 (0.319)	1.005 (0.064)	1.535 (0.186)
4.0	1.0	2.0	4.019 (0.409)	1.005 (0.051)	2.086 (0.295)
1.0	2.0	2.0	1.013 (0.113)	2.050 (0.265)	2.024 (0.174)
1.0	3.0	2.5	1.007 (0.119)	3.113 (0.439)	2.529 (0.215)
1.0	4.0	3.0	1.007 (0.131)	4.105 (0.606)	3.037 (0.258)

Table 2. Summary statistics for the survival times data set.

n	$\overline{X}$	s^2	b_1	b_2
72	99,819	6580,122	1,758	2,459

corresponding standard errors (SE) in parenthesis, for the four aforesaid distributions. For each model, the log-likelihood value is reported and the greatest value was obtained for the SPL distribution.

Table 3. Parameter estimates and log-likelihood values for L, PL, LS and SPL models for the survival times data set.

Parameters	L (SE)	PL (SE)	LS (SE)	SPL (SE)
$\widehat{\sigma}$	—	—	0.622 (0.651)	—
$\widehat{\theta}$	0.019 (0.001)	0.020 (0.007)	0.016 (0.016)	0.003 (0.001)
$\widehat{\alpha}$	—	0.992 (0.074)	—	1.562 (0.082)
$\widehat{q}$	—	—	3.673 (2.195)	1.900 (0.396)
Loglikelihood	−394.5197	−394.5185	−393.7586	**−389.7140**

Table 4. Akaike information criterion and Bayesian information criterion.

Criterion	L	PL	LS	SPL
AIC	791.039	793.037	793.517	**785.428**
BIC	793.316	797.590	800.347	**792.258**

In order to compare the distributions, we consider the usual Akaike information criterion (AIC) introduced by Akaike (1974) and Bayesian information criterion (BIC) proposed by Schwarz (1978). It is known that $AIC = 2k - 2\log \text{lik}$ and $BIC = k \log n - 2 \log \text{lik}$ where k is the number of parameters in the model, n is the sample size and log lik is the maximized value of the likelihood function. The model with lower values of AIC and/or BIC is preferred. Table 4 show the corresponding AIC and BIC for each model. For these data, AIC and BIC shows a better it of the SPL model. Figure 4 presents the histogram for the data with the fitted densities. Figure 5 depicts qqplots for each models. From these results, the SPL distribution provided a better fit than the other distributions under consideration.

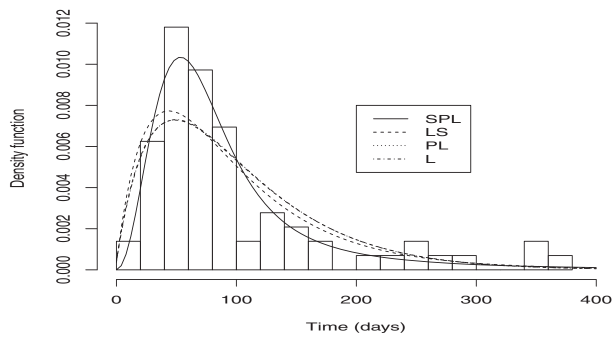

Figure 4. Models fitted by the maximum likelihood approach for survival time data set.

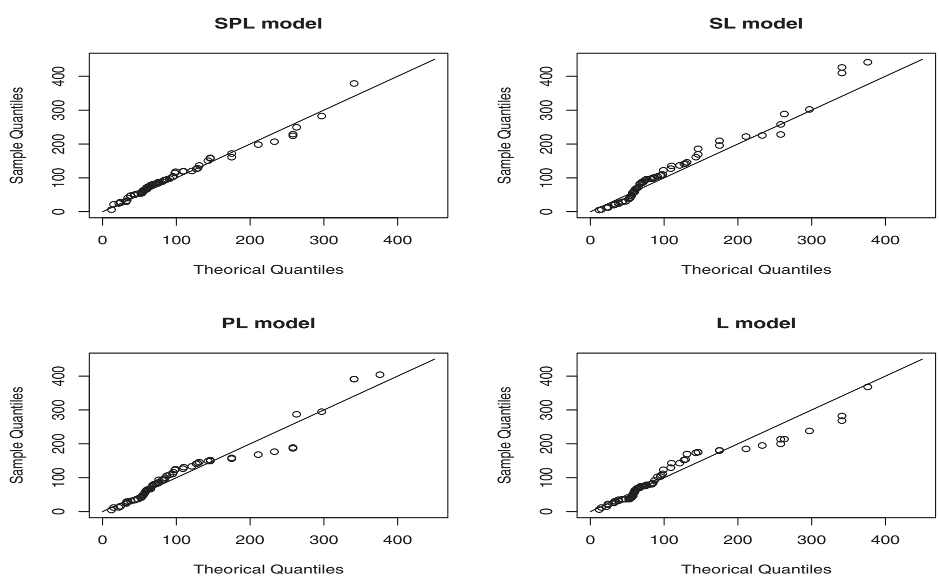

Figure 5. qqplots for the SPL, SL, PL and L models.

5. Concluding remarks

In this article we introduce an extension of the power Lindley distribution called the slashed power Lindley distribution. It arises from the ratio between two independent random variables, the power Lindley distribution in the numerator and a power of the uniform random variable in the denominator. The resulting model potentially has a larger kurtosis coefficient than the power Lindley distribution. Maximum likelihood estimators requires numerical procedures such as the Newton-Raphson algorithm. The derivation of the asymmetry and kurtosis coefficients illustrates the fact that the new distribution is able fit data sets for which the power Lindley distribution is adequate but with an excess of kurtosis. Application to real data has demonstrated that the proposed distribution can present better fit than distributions such as the Lindley, power Lindley and Lindley slash.

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