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To cite this article: Gino Chávez *et al* 2024 *J. Phys.: Conf. Ser.* **2701** 012105

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Semigroup theory and finite element method applied to a non-linear dissipative wave equation

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Abstract. We study the wave equation with a non-linear dissipative term associated to a bidimensional membrane with fixed boundary. We use the semigroup theory to consider the existence and uniqueness of solutions to the problem and we implement the finite element method to analyse the vibrating evolutionary equation. In particular we use Comsol Multiphysics software with a rectangular mesh to analyze the corresponding evolutionary system. We mention that this system can appear, for example, in the diaphragm of a centrifugal pump in mining processes.

1. Introduction

Over the years has been open the interest in the study of certain physical phenomena involved in our daily activities, such as sound, light, an earthquake, a magnetic field or how a radio works. There are similar phenomena at an industrial level, for example, in the operating principle of a spectrophotometer, associated with damped systems or with pumps and turbines, etc.

In this work we analyze the wave equation in two spatial dimensions with a non-linear dissipative term, modeling for example a square membrane fixed to its edges inside a centrifugal pump of water. We have two fundamental problems associated to this equation, the first one is a classical problem, which is known to have a unique solution using the semigroup theory; the second one represents the weak problem.

We analyze the numerical approximative solution for the weak problem using the finite element method (FEM) using a quadrangular mesh. The software chosen is the Comsol Multiphysics, software based in (FEM), easy to implement. Since this equation is time-dependent, this software gives the solution in a time interval and can also graphs the vibrations of the membrane considered as a evolutionary system.

The results show that, with a non-linear dissipative term and a rectangular mesh of finite elements, the vibrations of the membrane are decreasing until get steady in the time, and this can be verified with the software. In addition, we study the convergence of the solutions, showing that this property truly converges for a given range of time for the simulation.



2. The semilinear Cauchy problem

We will be focused in the analysis of the vibrations for a membrane given by the scalar function $u(x, y, t)$ which satisfies the equation

$$u_{tt} = \Delta u + g(u_t), \quad (1)$$

where $g(u_t)$ is a given a non-linear dissipative term and Δ is the corresponding Laplacian. We give also boundary and initial conditions, in a bounded spatial domain. The analogous problem, formulated in exterior domains with certain additional conditions is a very interesting an non-trivial one.

We can reformulate (1) in the form (we explicitly explain this below, see (7), (8) and (9))

$$\begin{cases} \frac{dU(t)}{dt} = AU(t) + G(U(t)), \\ U(0) = U_0, \end{cases} \quad (2)$$

where U is an functional vector, G is a non-linear operator, which depends on u_t and A is an infinitesimal generator of a C_0 -semigroup of contractions; U_0 represents a functional vector which provides the initial conditions to the problem.

Before (2), the non-homogeneous problem is given by

$$\begin{cases} \frac{dU(t)}{dt} = AU(t) + G(t) \\ U(0) = U_0. \end{cases} \quad (3)$$

The context of problem (3) is given by a Banach space X , a linear operator $A : D(A) \subseteq X \rightarrow X$ and a non-linear operator $G : [0, \infty) \rightarrow X$, with an associated initial condition $U_0 \in X$.

A classical solution for (3) is given by:

Definition 1. A classical solution for (3) is given by a function $U : [0, \infty) \rightarrow \mathbb{R}$ such that

$$U \in C([0, +\infty); D(A)) \cap C^1([0, +\infty); X)$$

which satisfies Cauchy problem.

We have:

Theorem 1. Let A the infinitesimal generator of a C_0 -semigroup $\{S(t)\}_{t \geq 0}$. For a function $G \in C^1([0, \infty); X)$ and an element $U_0 \in D(A)$ the Cauchy problem (3) has only one classical solution U given by:

$$U(t) = S(t)U_0 + \int_0^t S(t-s)G(s)ds. \quad (4)$$

For the semilinear problem (2) a generalized solution is given by:

Definition 2. A function $U \in C([0, +\infty), D(A))$, given by:

$$U(t) = S(t)U_0 + \int_0^t S(t-s)G(U(s))ds, \quad \forall t \geq 0 \quad (5)$$

is a generalized solution for problem (2), where A is the infinitesimal generator of a C_0 -semigroup of contractions $\{S(t)\}_{t \geq 0}$ acting on the Banach space X , and $G : X \rightarrow X$ is a non-linear operator with a initial value $U_0 \in X$.

We have in this case:

Theorem 2. *Let $G : X \rightarrow X$ globally Lipschitz with constant L . Let A be the infinitesimal generator of a C_0 -semigroup of contractions $\{S(t)\}_{t \geq 0}$ over X . Given $U_0 \in X$, there exists only one generalized solution U for the problem (2), that is, $U \in C([0, +\infty), X)$ and satisfies the following integral equation:*

$$U(t) = S(t)U_0 + \int_0^t S(t-s)G(U(s))ds, \quad \forall t \geq 0. \quad (6)$$

Furthermore, if $U(t)$ y $\widehat{U}(t)$ are two generalized solutions of (2), for two given initial values U_0 y \widehat{U}_0 respectively, then, for each $t \geq 0$ holds:

$$\|U(t) - \widehat{U}(t)\|_X \leq e^{Lt} \|U_0 - \widehat{U}_0\|_X.$$

Then, the mathematical model (1), with specified boundary and initial conditions is given by

$$\begin{cases} u_{tt} - \Delta u = g(u_t) & , \text{ en } \Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) ; u_t(x, 0) = u_1(x) & , \text{ en } \Omega, \\ u(x, t) = 0 & , \partial\Omega \times (0, +\infty), \end{cases} \quad (7)$$

where $\Omega \subset \mathbb{R}^n$ is an open set of class C^∞ with bounded boundary $\partial\Omega$.

All work well putting $U = \begin{pmatrix} u \\ u_t \end{pmatrix}$, $U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$, $G = \begin{pmatrix} 0 \\ g(u_t) \end{pmatrix}$ are elements of H defined by

$$H = H_0^1(\Omega) \times L^2(\Omega) \quad (8)$$

and $A : D(A) \subset H \rightarrow H$ is given by

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad (9)$$

with domain

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega). \quad (10)$$

The spaces $H^2(\Omega)$ and $H_0^1(\Omega)$ are the usual Sobolev spaces and $G : H \rightarrow H$ is a given non-linear operator.

It is also possible to prove the exponential decay of the solutions.

To see all the related theory in detail we recommend [1] and [2].

An interesting and non-trivial problem is to consider exterior problems with boundary conditions over dissipative term at infinity [3].

3. Variational formulation and finite element method

In this section we analyze the non-linear problem described above, using the finite element method. We focus on the case of quadrangular finite elements, modeling a certain domain to be discretized.

It is preferred to apply this method for those partial differential equations whose exact solutions are difficult to find. The basic thing is to replace the Hilbert space H on which its variational formulation is proposed by a finite-dimensional space V . With an appropriate division of the domain D the method is reduced to the resolution of a system of linear equations.

This method has several advantages over its similar ones, such as the finite difference method: it allows solving problems based on domains with a more complicated shape and allows solving nonlinear problems more easily.

The variational formulation for the problem posed works as follows: suppose that the vibration of the elastic membrane is described by the following two-dimensional equation

$$u_{tt} - \Delta u - g(u_t) = 0 \quad (x, y) \in \Omega, t \geq 0, \quad (11)$$

where g is the nonlinear dissipative term, with domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 : (x, y) \in (0, 1) \times (0, 1)\}. \quad (12)$$

In this case, Ω is an open set on \mathbb{R}^2 with boundary $\Gamma = \partial\Omega$.

We are going to consider the following initial and boundary conditions for the displacement and velocity, taking into account that the membrane must be fixed at its 4 ends.

$$u(x, y, 0) = u_0(x, y) \quad (x, y) \in \Omega \quad (13)$$

$$u_t(x, y, 0) = u_1(x, y) \quad (x, y) \in \Omega \quad (14)$$

$$u(x, y, t) = 0 \quad (x, y) \in \partial\Omega, t \geq 0 \quad (15)$$

As a first step, we are going to multiply the equation of the problem by a function v , which belongs to $H_0^1(\Omega)$, that is, to a Hilbert space that contains the functions that cancel each other at the border of the domain. It is then integrated over the domain on both sides of the equality.

We obtain

$$\int_{\Omega} u_{tt} v dx dy - \int_{\Omega} \Delta u v dx dy - \int_{\Omega} g(u_t) v dx dy = 0.$$

The second integral can be rewritten according to Green's theorem to give

$$\int_{\Omega} u_{tt} v dx dy - \left(\int_{\Omega} \frac{\partial u}{\partial n} v d\sigma - \int_{\Omega} \nabla u \nabla v \right) - \int_{\Omega} g(u_t) v dx dy = 0.$$

Because $v \in H_0^1(\Omega)$, we have for all $(x, y) \in \partial\Omega$ that $v = 0$.

It then follows

$$\int_{\Omega} u_{tt} v dx dy + \int_{\Omega} \nabla u \nabla v - \int_{\Omega} g(u_t) v dx dy = 0, \quad \forall v \in H_0^1(\Omega). \quad (16)$$

This last expression is the variational formulation of the posed problem. Furthermore, if to the expression $\int_{\Omega} u_{tt} v dx dy + \int_{\Omega} \nabla u \nabla v$ we call $B(u, v)$ and to $\int_{\Omega} g(u_t) v dx dy$ we referred by $l(v)$, then we have

$$B(u, v) = l(v)$$

This expression is the weak formulation of the considered problem, which has a unique solution according to the Lax-Milgram theorem. To solve the problem, the Finite Element Method is used [4, 5, 6], which consists of finding a function $u \in V$, such that $B(u, v) = l(v)$, where V is the space of all functions that vanish on the boundary of the considered domain.

The first thing to do is to take a domain Ω , which will be the region that will be discretized into smaller elements called finite elements

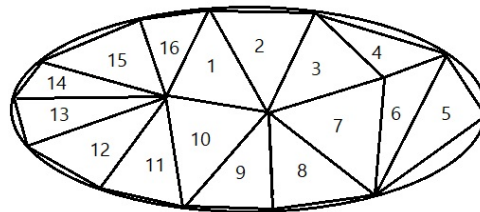


Figure 1: An example of a triangular discretization

Edges must be common between neighboring elements, so discretizations of Ω like this are not allowed

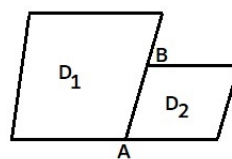


Figure 2: An example of a not allowed discretization

The problem is to construct appropriate base functions, which transform the problem to finding matrices K_{ij} and F_i such that

$$\sum K_{ij}x = \sum F_i,$$

where x will contain the numerical solutions to the problem. These matrices will be determined from the constructed base functions.

Applying the base functions to the initial problem, we have

$$K_{ij} = B(\phi_i, \phi_j) = \sum B^{(e)}(\phi_i^{(e)}, \phi_j^{(e)}) = \sum K_{ij}^{(e)},$$

$$F_i = l(\phi_i) = \sum l^{(e)}(\phi_i^{(e)}) = \sum F_i^{(e)}.$$

In this setting K_{ij} is known as *stiffness matrix* and F_i is the *load vector* del of the system. The *global* stiffness matrix K and the *global* load vector F are constructed, adding each local contribution in the corresponding places.

The previous system of equations is transformed to

$$K \cdot x = F. \tag{17}$$

This system, although linear, has a drawback: the global stiffness matrix K is singular, which makes it impossible to solve at the moment. To make K non-singular, boundary conditions must be incorporated, adding more individual contributions of K . Once the system is solved, the vector x provides the approximate solutions per *node*.

The domain Ω can also be discretized using triangular or quadrangular finite elements. In the case of keeping the idea of fixing nodes at the vertices of the element, then the simplest rectangular element will be one with 4 nodes. Now, it is about finding a function of two variables, which, evaluated at the corresponding values, results in the nodes of the rectangular element.

This function can be obtained from

$$p(x, y) = a_1 + a_2x + a_3y + a_4xy + a_5x^2 + a_6y^2 \dots$$

Since the basis functions can be determined with three nodal values, one of them is left over. However, quadratic functions require 6 nodal values, so determining the function would be quite complicated. This leads us to propose as a solution to retain the first 4 terms of $p(x, y)$. The first three terms are retained because they are non-quadratic, but among the following terms, it is chosen to retain the xy for reasons of better approximation to the exact solution.

In this way, the base function will be

$$p(x, y) = a_1 + a_2x + a_3y + a_4xy,$$

that is, a bilinear polynomial in x, y .

The procedure to determine the base functions in an element is analogous to that for triangles, with the difference that the systems for each node this time will have 4 equations and 4 variables.

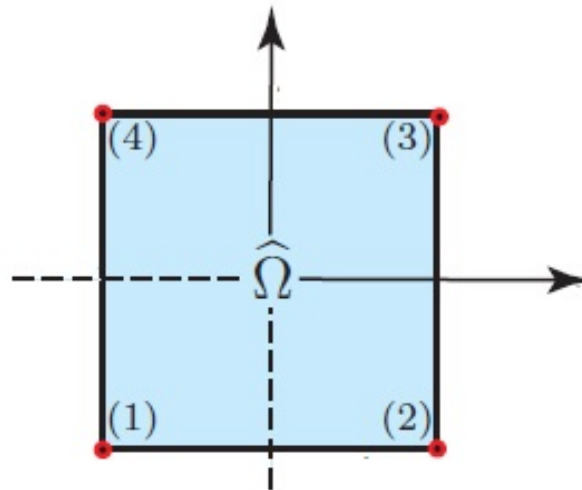


Figure 3: An example of a rectangular finite element

4. Determination of the stiffness matrix and the load vector

Let us consider a two-dimensional problem where the matrices K_{ij} of the weak form discretization are given by

$$K_{ij} = B(\phi_i, \phi_j) = \sum_{e=1}^E B^{(e)} \left(\varphi_i^{(e)}, \varphi_j^{(e)} \right) = \sum_{e=1}^E K_{ij}^{(e)},$$

where $K_{ij}^{(e)}$ are defined by

$$K_{ij}^{(e)} = \int_{D_e} \Delta \varphi_i^{(e)} \cdot \varphi_j^{(e)} dD \quad i, j = 1, 2, \dots, m.$$

It must be taken into account that the contributions to the elemental matrix of D_e are given only by the functions φ_i related to the nodes of the element. That is, for a triangular element, there would be $\varphi_i^{(e)}, \varphi_j^{(e)}, \varphi_k^{(e)}$, and the elementary matrix is made up of

$$K^{(e)} = \begin{bmatrix} K_{11}^{(e)} & K_{12}^{(e)} & K_{13}^{(e)} \\ K_{21}^{(e)} & K_{22}^{(e)} & K_{23}^{(e)} \\ K_{31}^{(e)} & K_{32}^{(e)} & K_{33}^{(e)} \end{bmatrix}, \quad (18)$$

where $K_{ij}^{(e)}$ are given by

$$K_{ij}^{(e)} = \int_D (J^{-1} \Delta_{\xi\eta} \varphi_i) \cdot (J^{-1} \Delta_{\xi\eta} \varphi_j) |J| d\xi d\eta \quad i, j = 1, 2, 3.$$

Here, J is the so-called Jacobian matrix, which contains the partial derivatives of a coordinate transformation ξ and η .

It must be taken into account that the stiffness matrix (18) is singular, so there would be problems solving the system of linear equations. However, by imposing the boundary conditions of the problem, this obstacle is overcome.

In the case of the load vector, we proceed in a similar way

$$F_i = l(\phi_i) = \sum_{e=1}^E l^{(e)}(\phi_i^{(e)}) \quad \text{con} \quad F_i^{(e)} = \int_{\Omega_e} f \phi_i^{(e)} d\Omega.$$

The elemental vector $F^{(e)} = [F_1^{(e)}, F_2^{(e)}, F_3^{(e)}]^T$ is therefore represented by

$$F_i^{(e)} = \int_{\hat{\Omega}} f \hat{\phi}_i |J| d\xi d\eta. \quad (19)$$

The stiffness matrix defined by (18) and the load vector defined by (19) define the respective system of linear equations, and its resolution will provide the numerical solutions of the problem (17).

5. Numerical simulation of the solution using Comsol Multiphysics

This section presents the numerical method used to simulate the displacements of a quadrangular elastic membrane, fixed at its 4 ends, and subject to a non-linear dissipation force, through the finite element method (FEM) [7, 8, 9].

Firstly, Comsol Multiphysics is presented as the program chosen to represent the solutions to the problem to be solved, then the variational formulation of the problem is presented, based on the information provided in the previous chapter, and finally the solutions obtained in the aforementioned program are shown, as well as their convergence.

A square membrane with side 1 and a material such that it allows vibrations when applying a non-linear external force is considered. The membrane is fixed at its 4 ends. It is assumed that, given the boundary conditions, the displacements are equal to zero on the contours. Meanwhile, the initial condition will be given by the following function of two variables

$$u(x, y) = 20xy(1 - x^2)(1 - y^2) \quad 0 < x < 1 \quad 0 < y < 1. \quad (20)$$

Since the membrane has no initial vibrations, $u_0(x, y) = 0$ and movement is forced at $t = 0$. A non-linear dissipation given by

$$g(u_t) = \arctan(u_t) \quad (21)$$

With these conditions, we will proceed to show the graphical solutions of the equation defined in (11) and the conditions defined in (13), (14) and (15), in addition to considering a rectangular mesh of finite elements.

Remembering that the equation is time dependent, solutions will be taken into account for different times between $t = 0$ and $t = 1$, both inclusive.

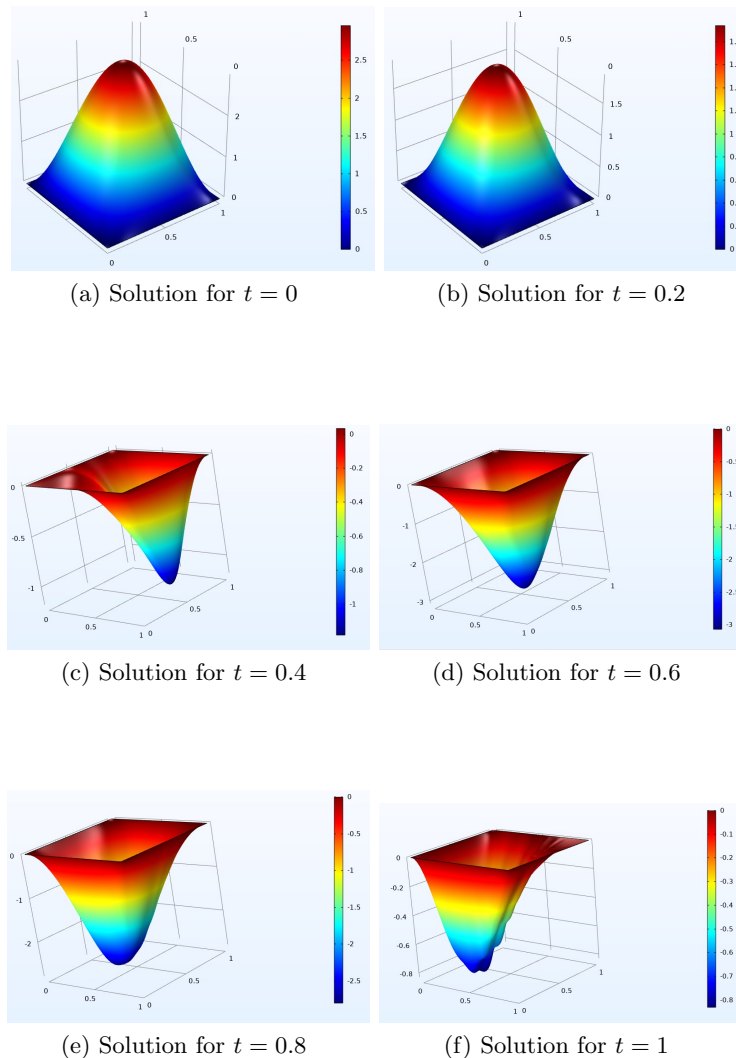


Figure 4: Approximate solutions for $u(x, y, t)$ of the wave equation with nonlinear dissipation using Comsol

The previous figures show that with a time of approximately 0 to 0.2 seconds the displacement $u(x, y, t)$ is stable, however after this time the displacements begin to decrease until they become approximately zero, and after 0.4 seconds these become negative. After 0.8 seconds the displacements decrease again, until the time of 1 second in which they once again approach 0. In effect, what the previous figures show is the dissipative effect of the force $g(u_t)$ in movements, so that these are reduced after one second has elapsed.

In addition, Comsol allows you to show the order of convergence of the solutions obtained. This is an important aspect in partial differential equations, since numerically solving any problem of this type implies approaching the theoretical solution, which means that the interest must always lie in minimizing the error, that is, the difference between the value calculated and the actual value. For the solved problem, Comsol graphs the convergence as follows

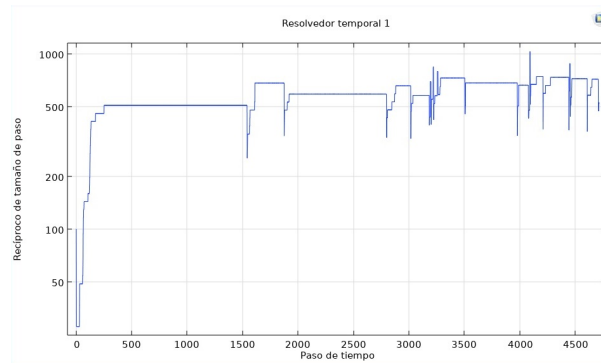


Figure 5: Convergence of the solution with triangular mesh

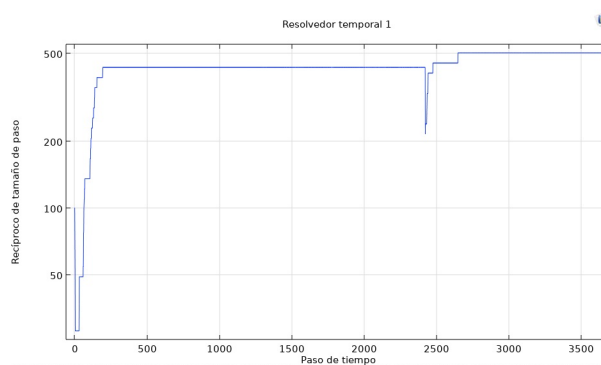


Figure 6: Convergence of the solution with rectangular mesh

Comsol will always plot the reciprocal of step size versus time step. Whatever the case, two things should be expected from this graph: that it increases and that it stabilizes over time. While in the triangular mesh graph there is precision but not accuracy of convergence, in the rectangular mesh graph we can verify the presence of both accuracy and precision, which provides greater reliability to the results obtained in the latter case.

Conclusions

In this work, we review in first place, in a very general way, the theory of semigroups and the semilinear Cauchy problem over which our problem is based.

Then, we use the Comsol program, formerly known as Femlab, to obtain the numerical solutions of the square membrane with side one, simulating a membrane of a centrifugal pump made of an elastic material that admits pressures of the fluid that circulating through it. The method in which all of the analysis is based is the Finite Element Method.

We formulate the approximative process using a quadrangular mesh. The graphics presented show the advantage in the use of quadrangular mesh over a triangular one, in the approximative process to the solution of the problem.

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