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CORRIGENDUM

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ABSTRACT
In [Martínez-Flórez G, Bolfarine H, Gómez HW. Skew-normal alpha-power model. Statistics. 2014;48(6):1414–1428] the authors make an error in calculating the information matrix of the skew-normal alpha-power model under the normality hypothesis $H_0 : (\lambda, \alpha) = (0, 1)$. Here we prove that the information matrix under H_0 is singular and we proposed a reparametrization that allows one to calculate the asymptotic distribution of the maximum likelihood estimator.

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1. Introduction

For $\phi_\lambda(z) = 2\phi(z)\Phi(\lambda z)$ and $\Phi_\lambda(z) = \int_{-\infty}^{z} \phi_\lambda(u)\,\mathrm{d}u$, Martínez-Flórez et al. [1] define the skew-normal alpha-power (PSN) distribution as follows:

$$f_X(x;\theta) = \alpha\eta^{-1}\phi_\lambda(z)\{\Phi_\lambda(z)\}^{\alpha-1}, \quad x \in \mathbb{R}, \tag{1}$$

where $z = (x - \xi)/\eta$, $\theta = (\xi, \eta, \lambda, \alpha)$, λ, α, ξ and η are the asymmetry, shape, location and scale parameters, respectively.

The purpose of this note is to correct the error that the authors make when calculating the information matrix $I_F(\theta)$ for $(\alpha, \lambda) = (1, 0)$ (conditions of symmetry, see p. 1423). The misuse of the numerical results of the elements of this matrix induces to erroneously calculate the determinant. It is easy to verify that the first and third columns of $I_F(\theta)$ are linearly dependent, this implies that the determinant is zero. Therefore, the conditions of regularity of the asymptotic theory for MLE are not met. Next, we will show that the information matrix under the assumption of symmetry is singular and we will propose a reparametrization that allows one to calculate the asymptotic distribution of the MLE.

2. Information matrix

Let $X_1, \dots, X_n$ be a random sample from the PSN distribution with localization and scale. The log-likelihood function for θ is $\sum_{1=1}^{n} l(\theta; X_i)$, where $l(\theta, x)$ is the log-likelihood for θ based on an observation x, that is,

$$l(\theta; x) = \log 2 + \log\alpha - \log\eta - z^2/2 + \log\Phi(\lambda z) + (\alpha - 1)\log\Phi_\lambda(z). \tag{2}$$

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The score function is $\sum_{i=1}^{n} S_\theta(\theta, X_i)$, where $S_\theta(\theta, x) = \partial l(\theta; x)/\partial\theta$ is the vector $(S_\xi, S_\eta, S_\lambda, S_\alpha)$ whose elements are

$$S_\xi = -\frac{1}{\eta}\left\{-z + \lambda\frac{\phi(\lambda z)}{\Phi(\lambda z)} + (\alpha - 1)\frac{\phi_\lambda(z)}{\Phi_\lambda(z)}\right\},$$

$$S_\eta = -\frac{1}{\eta}\left\{1 - z^2 + \lambda z\frac{\phi(\lambda z)}{\Phi(\lambda z)} + (\alpha - 1)z\frac{\phi_\lambda(z)}{\Phi_\lambda(z)}\right\},$$

$$S_\lambda = z\frac{\phi(\lambda z)}{\Phi(\lambda z)} - \sqrt{2/\pi}\frac{(\alpha-1)}{1+\lambda^2}\frac{\phi(\sqrt{1+\lambda^2}z)}{\Phi_\lambda(z)},$$

$$S_\alpha = \frac{1}{\alpha} + \log\Phi_\lambda(z).$$

Here we are interested in knowing what happens to these score equations when the symmetric subfamily is obtained, that is, what happens under the normality hypothesis $H_0 : (\lambda, \alpha) = (0, 1)$. If this occurs, the above equations become $S_\xi^* = z/\eta$, $S_\eta^* = (z^2 - 1)/\eta$, $S_\lambda^* = \sqrt{2/\pi}\, z$ and $S_\alpha^* = 1 + \log\Phi(z)$, where $S_\theta^* = (S_\xi, S_\eta, S_\lambda, S_\alpha)|_{(\lambda,\alpha)=(0,1)}$. Here we see that S_λ^* is linearly dependent on S_ξ^* as in the skew-normal model of Azzalini [2]. As is known, this leads to the information matrix being singular under H_0. Therefore, the determinant given by $|I_F(\theta)| = 0$, where

$$I_F(\theta) = n\begin{pmatrix} \frac{1}{\eta^2} & 0 & \frac{1}{\eta}\sqrt{\frac{2}{\pi}} & \frac{a_1}{\eta} \\ 0 & \frac{2}{\eta^2} & 0 & \frac{a_2 - a_0}{\eta} \\ \frac{1}{\eta}\sqrt{\frac{2}{\pi}} & 0 & \frac{2}{\pi} & a_1\sqrt{\frac{2}{\pi}} \\ \frac{a_1}{\eta} & \frac{a_2 - a_0}{\eta} & a_1\sqrt{\frac{2}{\pi}} & 1 \end{pmatrix}$$

and $a_r = \int_0^1 (\Phi^{-1}(u))^r (1 + \log u)\, \mathrm{d}u$ for $r = 0,1,2$.

Note from this matrix that the columns corresponding to the parameters ξ and λ are linearly dependent, implying that it is a singular matrix, and hence, regularity conditions are not satisfied. One consequence of this fact is that likelihood ratio statistics for testing normality are no longer distributed according to the central chi-square distribution (see [3]). This irregularity was discussed by Azzalini [2] in the context of the SN model, and was later studied systematically by Chiogna [4]. Arellano-Valle and Azzalini [3] discuss the singularity of the multivariate skew-normal model using centred reparametrization, DiCiccio and Monti [5] also studied this singularity in the context of skew-exponential power distribution. Salinas et al. [6] in the context of the extended skew-exponential power distribution, and similar to Chiogna [4], used the methodology of Rotnitzky et al. [7] to obtain an appropriate reparametrization from which it is possible to calculate the asymptotic distribution of the MLE.

As in DiCiccio and Monti [5], we make use of Theorem 3 from Rotnitzky et al. [7] to derive a reparametrization for model (1) and provide a solution to the singularity problem for $\boldsymbol{\theta} = \boldsymbol{\theta}^* = (\xi^*, \eta^*, \lambda^*, \alpha^*)$ where $(\lambda^*, \alpha^*) = (0, 1)$. We start by denoting $S_j(\theta) = \partial l(\theta; X)/\partial\theta_j$, $1 \le j \le 4$ as the score with respect to θ_j and $S_j = S_j(\boldsymbol{\theta}^*)$. Using the first derivatives of Equation (2), we can compute the scores S_1, S_2, S_3 and S_4. In fact, $S_1 = Z^*/\eta^*$, $S_2 = (Z^{*2} - 1)/\eta^*$, $S_3 = \sqrt{2/\pi}\, Z^*$ and $S_4 = 1 + \log\Phi(Z^*)$, where $Z^* = (X - \xi^*)/\eta^*$. It is then easy to verify that $S_3 = \eta^*\sqrt{2/\pi}\, S_1$ so that there is a 1×3 vector $\boldsymbol{K}$ such that $S_3 = \boldsymbol{K}(S_1, S_2, S_4)'$ and $\boldsymbol{K} = (\eta^*\sqrt{2/\pi}, 0, 0)$. This shows that the vectors S_1, S_2 and S_4 are linearly independent.

Table 1. Approximations for the values a_r with $r=0,1,2,3$.

a_r	Value	Absolute error
$r=0$	−1.711559e−16	< 9e−16
$r=1$	0.903197	< 2.8e−05
$r=2$	−0.595636	< 2.6e−07
$r=3$	2.909863	< 1.8e−405

We apply now the iterative reparametrization process and the results of Theorem 3 (see [7]) to derive the asymptotic distribution of the MLE for model (1). As seen above, the individual contribution of the first derivatives of the log-likelihood in Equation (2) evaluated at $\boldsymbol{\theta} = \boldsymbol{\theta}^*$, (S_1, S_2, S_3, S_4), is given by $[Z^*/\eta^*, (Z^{*2}-1)/\eta^*, \sqrt{2/\pi}Z^*, 1+\log\Phi(Z^*)]$. Letting $S_3 = K_{11}S_1$, where $K_{11} = \eta^*\sqrt{2/\pi}$, S_1, S_2 and S_4 are linearly independent, so that we consider the reparametrization $\widetilde{\boldsymbol{\theta}} = (\widetilde{\xi}, \eta, \lambda, \alpha)$ with $\widetilde{\xi} = \xi + K_{11}\lambda$. Therefore, the pdf in Equation (1) can be rewritten as

$$f_X(x;\widetilde{\boldsymbol{\theta}}) = \alpha\eta^{-1}\phi_\lambda(z)\{\Phi_\lambda(z)\}^{\alpha-1}, \quad x \in \mathbb{R}, \tag{3}$$

where $z = (x - \widetilde{\xi} + K_{11}\lambda)/\eta$. The second partial derivative with respect to λ of the log-likelihood function (3) evaluated at $\boldsymbol{\theta} = \boldsymbol{\theta}^*$ is equal to $\widetilde{S}_3^{(2)} = 2(1 - Z^{*2})/\pi$. Clearly, $\widetilde{S}_3^{(2)}$ can be written as $\widetilde{S}_3^{(2)} = K_{22}S_2$, where $K_{22} = -2\eta^*/\pi$. Thus, it is necessary to do another reparametrization where $\widetilde{\xi} = \xi + K_{11}\lambda$ and $\widetilde{\eta} = \eta + 2^{-1}K_{22}\lambda^2$. This new reparametrization affects the location-scale parameters so that $z = (x - \widetilde{\xi} + K_{11}\lambda)/(\widetilde{\eta} - 2^{-1}K_{22}\lambda^2)$ in Equation (3). We now compute the third partial derivative with respect to λ of the new log-likelihood evaluate at $\boldsymbol{\theta} = \boldsymbol{\theta}^*$ leading to $\widetilde{S}_3^{(3)} = 2^{1/2}((4-\pi)Z^{*3} - 6Z^*)/\pi^{3/2}$. Given that $\widetilde{S}_3^{(3)}$ is a function of Z^{*3}, it cannot be a linear combination of S_1, S_2 and S_4. Therefore, the iterative reparametrization procedure is stopped, and by Theorem 3 in Rotnitzky et al. [7]

$$[n^{1/2}(\widehat{\xi} - \xi^* + \widehat{\eta}\widehat{\lambda}(2/\pi)^{1/2}), \quad n^{1/2}(\widehat{\eta} - \eta^* - \widehat{\eta}\widehat{\lambda}^2/\pi), \quad n^{1/6}\widehat{\lambda}, n^{1/2}(\widehat{\alpha} - 1)]$$

converges, under the hypothesis $\boldsymbol{\theta} = \boldsymbol{\theta}^*$, to a vector $(Z_1, Z_2, Z_3^{1/3}, Z_4)$ where (Z_1, Z_2, Z_3, Z_4) is a random normal vector with null mean vector and covariance matrix equal to the inverse of the covariance matrix of the vector $(S_1, S_2, \widetilde{S}_3^{(3)}/3!, S_4)$, which is given by

$$n^{-1}\begin{pmatrix} \frac{1}{\eta^2} & 0 & \frac{2-\pi}{\eta\sqrt{2\pi^3}} & \frac{a_1}{\eta} \\ 0 & \frac{2}{\eta^2} & 0 & \frac{a_2 - a_0}{\eta} \\ \frac{2-\pi}{\eta\sqrt{2\pi^3}} & 0 & \frac{5\pi^2 - 28\pi + 44}{6\pi^3} & \frac{(4-\pi)a_3 - 6a_1}{\sqrt{18\pi^3}} \\ \frac{a_1}{\eta} & \frac{a_2 - a_0}{\eta} & \frac{(4-\pi)a_3 - 6a_1}{\sqrt{18\pi^3}} & 1 \end{pmatrix}^{-1}.$$

Using the `integrate` function of the `stats` package of R [8], we can numerically calculate the values of a_r for $r=0,1,2,3$. Table 1 shows the approximations for these quantities .

Disclosure statement

No potential conflict of interest was reported by the authors.

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