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E. Morales-Campaña ; P. Martin

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E. Morales-Campaña[a] and P. Martin[b]

AFFILIATIONS

Departamento de Física, Facultad de Ciencias Básicas, Universidad de Antofagasta, Antofagasta, Chile

[a]Author to whom correspondence should be addressed: esthefano.morales.campana@uantof.cl
[b]Electronic mail: pablo.martin@uantof.cl. This author is a fellow of the International Association of Advanced Materials.

ABSTRACT

A new approximation method for the plasma dispersion function $Z(\zeta)$ is presented. Multipoint quasi-rational approximation technique is used to find a bridge function that connects the power series and the asymptotic expansion of the function $Z(\zeta)$ using rational functions combined with exponential functions. An approximation with a polynomial of degree 10 is performed for the function $Z(\zeta)$, and the results obtained are compared with those of previous approximations from the literature. The results of this approximation were a relative error of $\varepsilon = 0.0035$ for $\mathrm{Re}[\tilde{Z}(\zeta)]$ and a relative error of $\varepsilon = 0.0011$ for $\mathrm{Im}[\tilde{Z}(\zeta)]$, which are lower than those of the other existing approximations.

I. INTRODUCTION

In the theory of linearized waves or oscillations in high-temperature plasmas, with or without a magnetic field, a function with a complex argument called the plasma dispersion function, $Z(\zeta)$, is determined. This function originates with Landau's treatment of electrostatic waves in plasmas in 1946.[1] The Fried and Conte plasma dispersion function is defined through the Hilbert transform of the Gaussian.[2] Through the years, different studies on the plasma dispersion function have been carried out to obtain a simple and easy-to-use approximation using different treatments and computational techniques.[3–17] This function is important to determine the damping and growth of plasma waves as well as instabilities, and it is also relevant for the theoretical and experimental applications in physics.[18–31]

In this work, new approximations have been determined for $Z(\zeta)$, using new techniques, as the "multipoint quasirational approximations," MPQAs. In this method, the terms of the power series and the asymptotic expansion are used. In this technique, a power series and asymptotic expansion are both used to determine a bridge function connecting both expansions; in this way, the technique is different from the Padé method,[16] where only power series is used. This procedure approaches different research areas with good results.[32–39] Previous treatment with this method was found by Martin *et al.*[4] with a four pole rational function, denoted by $Z_{53}(\zeta)$, where the maximum relative errors for the real part of $Z_{53}(\zeta)$ were 0.005 and those for the real part of $Z'_{53}(\zeta)$ were 0.035. However, the results were only determined for only a part of the complex plane as we will show in Sec. II. This approximation was better, in general, than those obtained by other authors. However, some applications mainly in the instability zones require higher accuracy for $Z(\zeta)$ and $Z'(\zeta)$ than those published until now. For this reason, a new approximation with a quotient of polynomials of ten degrees has been determined here. The best results were found using five terms of the power series, four terms of the asymptotic expansion, and an intermediate term as a condition, obtaining a relative error in the real part of 0.0035 and in the imaginary part of 0.0011. Furthermore, there is an excellent concordance for all the values of ζ, mainly in the superior plane as it is usually required. Furthermore, the multipolar quasirational approximation obtained here preserves the main symmetries of $Z(\zeta)$, $Z(\zeta^*) = -[Z(-\zeta)]^*$.

A general procedure for obtaining the values of rational functions is described in Sec. II. The results are presented in the graphical form in Sec. III. The analysis and discussion of the different approximations are done in Sec. IV, and Sec. V is dedicated to the conclusion.

II. GENERAL PROCEDURE

Rational functions will be used together with an exponential function. The parameters of the approximation will be determined by equalizing terms with the power series and the asymptotic expansion, with the terms of an approximate function. Considering the plasma dispersion function $Z(\zeta)$, the approximated function is written in the form

$$\tilde{Z}(\zeta) = -2\zeta \frac{\sum_{i=0}^{n} p_i \zeta^i}{\left(1 + \sum_{j=1}^{n} q_j \zeta^j\right)} + i\sqrt{\pi} e^{-\zeta^2}, \tag{1}$$

where $\zeta = x + iy$. Here, the numerator of the polynomial is chosen one degree less than the denominator to obtain the required asymptotic behavior in $O(1/\zeta)$. This is also the reason of the initial factor ζ. The parameters to be determined are p_i $(i = 0, \ldots, 4)$ and q_j $(j = 1, \ldots, 5)$. The approximate function $\tilde{Z}(\zeta)$ is written as

$$\tilde{Z}(\zeta) = -2\zeta \frac{p_0 + p_1\zeta^2 + p_2\zeta^4 + p_3\zeta^6 + p_4\zeta^8}{1 + q_1\zeta^2 + q_2\zeta^4 + q_3\zeta^6 + q_4\zeta^4 + q_5\zeta^{10}} + i\sqrt{\pi} e^{-\zeta^2}. \tag{2}$$

For the MPQA method, the exponential term does not have the value of σ since it is not required.

The power series is written as

$$Z(\zeta) = i\sqrt{\pi} e^{-\zeta^2} - 2\zeta\left[1 - \frac{2}{3}\zeta^2 + \frac{4}{15}\zeta^4 - \frac{8}{105}\zeta^6 + \frac{16}{945}\zeta^8 - \frac{32}{10\,395}\zeta^{10} + \cdots\right], \tag{3}$$

and the asymptotic expansion is given by

$$Z(\zeta) = i\sigma\sqrt{\pi} e^{-\zeta^2} - \frac{1}{\zeta}\left[1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \frac{15}{8\zeta^6} + \frac{105}{16\zeta^8} + \frac{945}{32\zeta^{10}} + \cdots\right], \tag{4}$$

where σ has values

$$\sigma \equiv \begin{cases} 0, & \mathrm{Im}\{\zeta\} > 0, \\ 1, & \mathrm{Im}\{\zeta\} = 0, \\ 2, & \mathrm{Im}\{\zeta\} < 0. \end{cases} \tag{5}$$

The treatment for the function $\tilde{Z}(\zeta)$ is developed with the power series and the asymptotic expansion excluding the exponential term of each function. In this way, the values of p_i and q_j can be obtained. It is necessary to rationalize the equations to avoid non-linear equations; in this way, only linear equations are obtained. For the power series, equating the terms of the series with the function $\tilde{Z}$, we obtain

$$p_0 + p_1\zeta^2 + p_2\zeta^4 + p_3\zeta^6 + p_4\zeta^8 = (1 + q_1\zeta^2 + q_2\zeta^4 + q_3\zeta^6 + q_4\zeta^4 + q_5\zeta^{10})\left(1 - \frac{2}{3}\zeta^2 + \frac{4}{15}\zeta^4 - \frac{8}{105}\zeta^6 + \frac{16}{945}\zeta^8 - \frac{32}{10\,395}\zeta^{10} + \cdots\right), \tag{6}$$

and finally, the equations will be now written as

$$\begin{aligned} p_0 &= 1, \\ p_1 &= q_1 - 2/3, \\ p_2 &= q_2 - (2/3)q_1 + 4/15, \\ p_3 &= q_3 - (2/3)q_2 + (4/15)q_1 - 8/105, \\ p_4 &= q_4 - (2/3)q_3 + (4/15)q_2 - (8/105)q_1 + 16/945, \\ 0 &= q_5 - (2/3)q_4 + (4/15)q_3 - (8/105)q_2 + (16/945)q_1 - 32/10\,395. \end{aligned} \tag{7}$$

For the asymptotic expansion, equaling terms with the function $\tilde{Z}$,

$$-2\zeta \frac{p_0 + p_1\zeta^2 + p_2\zeta^4 + p_3\zeta^6 + p_4\zeta^8}{1 + q_1\zeta^2 + q_2\zeta^4 + q_3\zeta^6 + q_4\zeta^4 + q_5\zeta^{10}} = -\frac{1}{\zeta}[1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \frac{15}{8\zeta^6} + \frac{105}{16\zeta^8} + \frac{945}{32\zeta^{10}} + \cdots], \tag{8}$$

once a simple algebra is done,

$$\frac{2p_0}{\zeta^8} + \frac{2p_1}{\zeta^6} + \frac{2p_2}{\zeta^4} + \frac{2p_3}{\zeta^2} + 2p_4 = \left(\frac{1}{\zeta^{10}} + \frac{q_1}{\zeta^8} + \frac{q_2}{\zeta^6} + \frac{q_3}{\zeta^4} + \frac{q_4}{\zeta^2} + q_5\right) \times \left(1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \frac{15}{8\zeta^6} + \frac{105}{16\zeta^8} + \frac{945}{32\zeta^{10}} + \cdots\right), \tag{9}$$

and we obtain

$$\begin{aligned} 2p_1 &= q_2 + (1/2)q_3 + (3/4)q_4 + (15/8)q_5, \\ 2p_2 &= q_3 + (1/2)q_4 + (3/4)q_5, \\ 2p_3 &= q_4 + (1/2)q_5, \\ 2p_4 &= q_5. \end{aligned} \tag{10}$$

With this method, Eqs. (7) and (10) are used. From these ten equations, their ten parameters that are necessary for $\tilde{Z}(\zeta)$ are obtained. The parameters are obtained using a system of linear equations or by a 10×10 matrix. It is observed that when evaluating $\tilde{Z}(\zeta)$ with the values found, a relative error that is not so good is obtained. Therefore, a condition is required to improve the relative error. For lambda, we will follow the procedure of the latest publications on MPQA technique, which consists of leaving a parameter free, minimizing the maximum relative error. This parameter will be denoted by λ, and it will replace the parameter q_5; the approximation will be written now as λ_0. The approximation and equations are as follows:

$$\tilde{Z}(\zeta) = -2\zeta \frac{p_0 + p_1\zeta^2 + p_2\zeta^4 + p_3\zeta^6 + p_4\zeta^8}{1 + q_1\zeta^2 + q_2\zeta^4 + q_3\zeta^6 + q_4\zeta^4 + \lambda\zeta^{10}} + i\sqrt{\pi}e^{-\zeta^2}. \tag{11}$$

The equations become

$$\begin{aligned} p_0 &= 1, \\ p_1 &= q_1 - 2/3, \\ p_2 &= q_2 - (2/3)q_1 + 4/15, \\ p_3 &= q_3 - (2/3)q_2 + (4/15)q_1 - 8/105, \\ p_4 &= q_4 - (2/3)q_3 + (4/15)q_2 - (8/105)q_1 + 16/945, \end{aligned} \tag{12}$$

where the last equation of Eq. (7) will not be considered. The equations can be written as

$$\begin{aligned} 2p_1 &= q_2 + (1/2)q_3 + (3/4)q_4 + (15/8)\lambda, \\ 2p_2 &= q_3 + (1/2)q_4 + (3/4)\lambda, \\ 2p_3 &= q_4 + (1/2)\lambda, \\ 2p_4 &= \lambda. \end{aligned} \tag{13}$$

If an approximation is made for polynomials of degrees 2, 4, 6, 8, 10, 12, and others, the value of lambda will not be the same since it is adjusted to each approximation made. In the case of our research, the value of lambda does not change because the approximation is only for polynomials of degree 10. The value of the parameter λ is obtained by comparing the function $\tilde{Z}(\zeta)$ with the exact function, of all the possible values of λ, the one that best fits the exact function $Z(\zeta)$ is chosen. The value of λ with the minimum of the maximum relative errors will be taken as the best one, and it will denoted as λ_0. The value of λ_0 is 0.006 517, and the nine parameters are as follows:

$$\begin{aligned} p_0 &= 1, \\ p1 &= (1/144)(32 + 945 \cdot \lambda_0), \\ p_2 &= (1/7560)(416 + 23\,635 \cdot \lambda_0), \\ p_3 &= (1/3780)(32 + 4725 \cdot \lambda_0), \\ p_4 &= 0.006\,517/2, \\ q_1 &= (1/144)(128 + 945 \cdot \lambda_0), \\ q_2 &= (1/42)(16 + 315 \cdot \lambda_0), \\ q_3 &= (1/630)(64 + 2835 \cdot \lambda_0), \\ q_4 &= (2/945)(8 + 945 \cdot \lambda_0). \end{aligned} \tag{14}$$

III. RESULTS

In Table I, the results of p_0, p_1, p_2, p_3, p_4, q_1, q_2, q_3, and q_4 are shown. In addition, we have the value of $\lambda_0 = 0.006\,517$. The values of the nine parameters p_i and q_j were found, and the value of λ was obtained; now, it is possible to graph the curves of $\tilde{Z}(\zeta)$. The relative error of the function $\tilde{Z}(\zeta)$ is found, and the maximum relative errors for the real part and those for the imaginary part are $\varepsilon = 0.0035$ and $\varepsilon = 0.0011$.

In Fig. 1, the MPQAs compared to the approximations of Martin and Pécseli are shown. The differences that exist between the approximations are small; the curves shown in Fig. 1 are not appreciated. Therefore, the maximum relative errors are obtained for each approximation; in this way, we can observe the differences that exist between them. Only in the case of the approximation of $Z(\zeta)$, there is similarity in the curves; for its derivative, there are important differences. The approximation carried out by Pécseli[9] is characterized by being a function that is evaluated in sections, that is, for each section that is developed, there is a different equation. This approximation has four intervals on the x axis. The intervals are $Z^2(x) < 6.25$, $6.25 < Z^2(x) < 12.25$, $12.25 < Z^2(x) < 25$, and $Z^2(x) > 25$. When comparing the approximation with the exact function $Z(\zeta)$, the maximum relative error for the real and imaginary part is 0.002 and 0.007, respectively. In the approximation of Martin *et al.*,[4] the function found is a polynomial of degree 4 and is called Z_{53}. The function Z_{53} is a rational function and is composed of poles and residues. Comparing the approximation with the exact function $Z(\zeta)$, it is found that the maximum relative error for the real part is 0.005 and that for the imaginary part is 0.02. For the MPQA, a relative error similar to the other approximations is obtained; the maximum

TABLE I. Values of the parameters p_i, q_j, and λ_0.

Value of λ_0	Parameters of p_i and q_j
0.006 517	$p_0 = 1.0$ $p1 = 0.264\,990\,1$ $p_2 = 0.075\,401\,0$ $p_3 = 0.016\,611\,8$ $p_4 = 0.003\,258\,48$ $q_1 = 0.932\,003\,4$ $q_2 = 0.429\,500\,3$ $q_3 = 0.132\,003\,4$ $q_4 = 0.030\,034\,2$

relative error for the real and imaginary part is 0.0035 and 0.0011, respectively. It is interesting to show the four curves; however, the differences between the approximations of the real and imaginary curves of $Z(\zeta)$ are so small that no difference can be shown on this scale.

Figure 2 shows a peak, which is the maximum error obtained. The relative error of the imaginary part of Z is obtained for the range $0 \le x \le 6$. In Fig. 3, the relative error of the part $\mathrm{Re}[Z(\zeta)]$ in the range of $0 \le x \le 6$ is shown. There are two peaks of maximum error; the maximum value is 0.0035.

To obtain the absolute error of the approximations, a difference is made between the Martin and MPQA; the values of $Z_{app} - Z$ were obtained. Z_{app} are the values of the approximations to be compared, and Z is the exact plasma dispersion function. The complex map of the absolute error of the approximations is shown in Fig. 4. The values of the imaginary part of $Z_{app} - Z$ are shown on the ordinate axis, and the values of the real part of $Z_{app} - Z$ are shown on the abscissa axis. It can be seen that the absolute error of the Martin approximation is shown with a

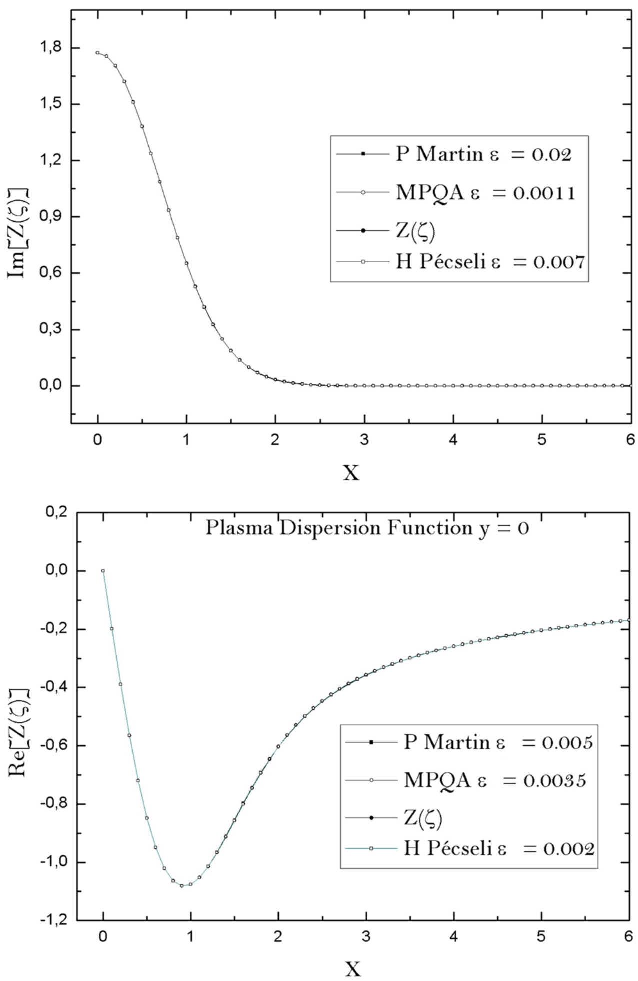

FIG. 1. The approximations of Martin and Pécseli and MPQA are shown with their respective relative errors for $Z(\zeta)$.

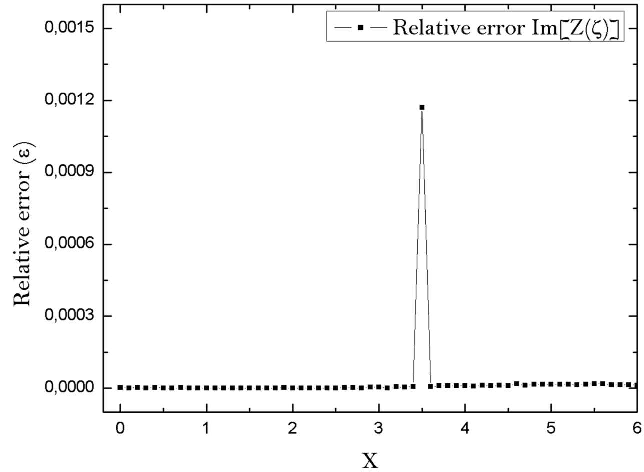

FIG. 2. Relative error of the imaginary part of $\tilde{Z}(\zeta)$ for $y = 0$. The maximum relative error is 0.0011.

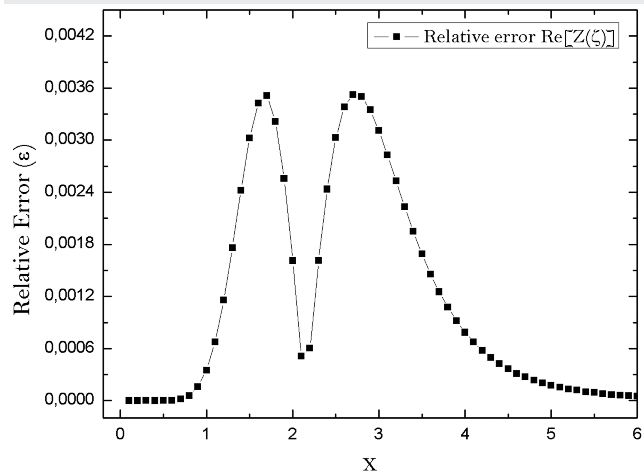

FIG. 3. Relative error of the real part of $\tilde{Z}(\zeta)$ for $y = 0$. The maximum relative error is 0.0035.

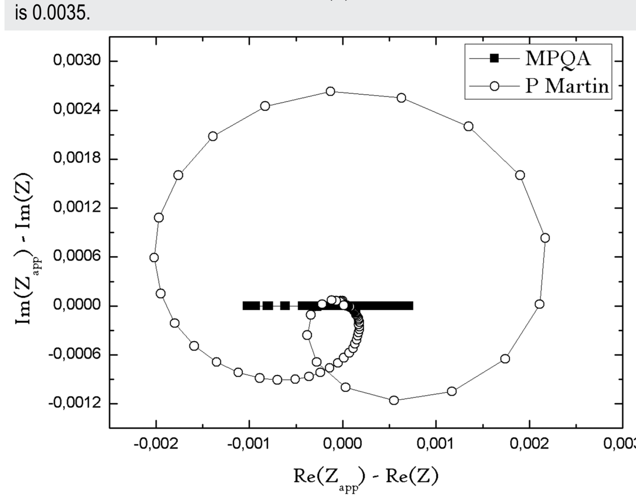

FIG. 4. Comparison of the complex map of Martin approximation and MPQA with the exact function $Z(\zeta)$.

TABLE II. Values of the maximum relative errors of each approximation for $Z(\zeta)$ and $Z'(\zeta)$.

Approximations	Maximum relative errors (real)	Maximum relative errors (imaginary)
MPQA $\tilde{Z}(\zeta)$	0.0035	0.0011
Martin $Z(\zeta)$	0.050	0.020
Pécseli $Z(\zeta)$	0.0020	0.0070
MPQA $\tilde{Z}'(\zeta)$	0.026	0.000 018
Martin $Z'(\zeta)$	0.035	0.11
Pécseli $Z'(\zeta)$	0.32	0.46

dotted line and the MPQA is shown with a square line. The absolute error of MPQA is less compared to that of other approaches; the absolute error of $\mathrm{Im}(Z_{app}) - \mathrm{Im}(Z)$ is less than 10^{-25}. There are other methods in the literature that approximate 4, 8, 12, 16 up to 24 degrees of polynomials.[19–21] These approximations are made with the Padé method using only the power series with an absolute error of 10^{-13}.

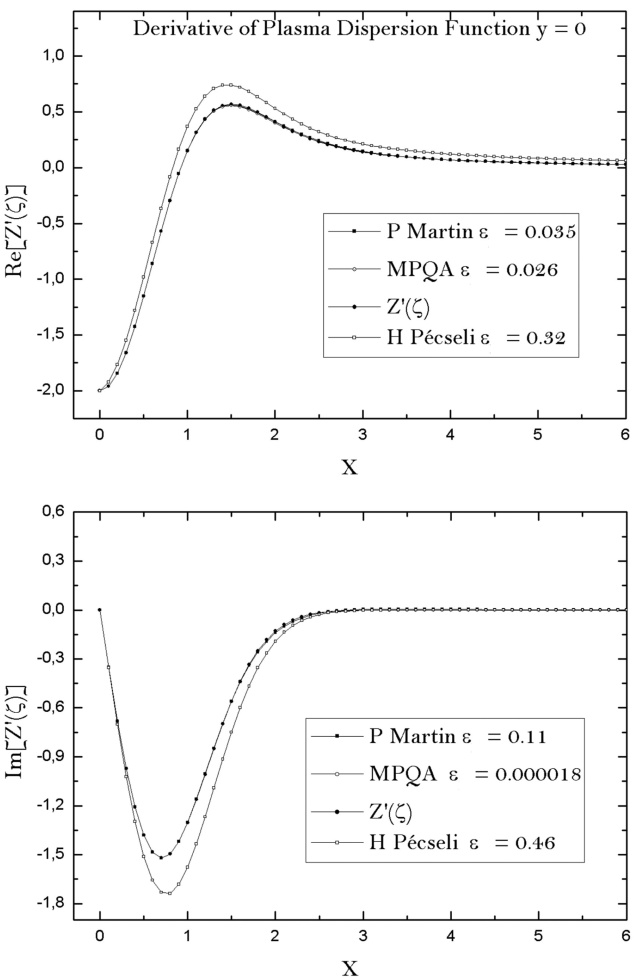

FIG. 5. The real and imaginary part of the derivative of the approximations compared to the exact function.

To obtain the derivative of the function $\tilde{Z}(\zeta)$, we have the following equation:

$$Z'(\zeta) = -2[1 + \zeta Z(\zeta)]. \tag{15}$$

To obtain $\tilde{Z}'(\zeta)$, we must again use the values of p_i and q_j; these are found in Table I with the condition λ_0. Table II shows the values of the maximum relative errors for each approximation for the real and imaginary part of $Z(\zeta)$ and $Z'(\zeta)$.

Using the values from Table I in the MPQA, in Eq. (2), it is possible to obtain the curves of $\tilde{Z}'(\zeta)$, with $\zeta = x + iy$, for $y = 0$. The function $Z'(\zeta)$ is obtained for the approximations of Pécseli and Martin and MPQA. Table II shows the maximum relative errors of the three approximations with respect to the exact function. Both the function Z and its derivative Z' are presented.

Figure 5 shows the real part of the approximations $Z'(\zeta)$, where the maximum relative error of the Martin approximation is obtained with a maximum relative error of $\varepsilon = 0.035$. In addition, we have the MPQA where a maximum relative error of $\varepsilon = 0.026$ is obtained and the Pécseli approximation has a relative error of $\varepsilon = 0.32$. The lower image of Fig. 5 shows the approximate curves of the imaginary part of $Z'(\zeta)$, where $\zeta = x + iy$ with $y = 0$. We have Martin's approximation with a relative error of $\varepsilon = 0.11$. The MPQA is shown with a relative error of $\varepsilon = 0.000\,018$, and the Pécseli approximation is shown with a relative error of $\varepsilon = 0.46$. It is clear that the approximation presented here for $Z'(\zeta)$ is much better than the previous approximations of Martin and Pécseli.

IV. ANALYSIS AND DISCUSSION

This research focuses on discussing the method of the multipolar quasirational approximation (MPQA) applied to the plasma dispersion function, comparing it with other existing methods in the literature. This method (MPQA) was tested for two, four, six, and eight degrees of polynomials, so it does not give a good result for this type of approach. Therefore, it was decided to make a ten degree approximation to improve the fit of the curve. The first comparison is made with the multipolar method of Martin since this approximation has been the most used in different investigations at an international level and is applied in different areas of physics. Martin's method uses the power series and asymptotic expansion to find the poles and residuals. This method is quite useful and relatively easy to get the values of the $Z(\zeta)$ function. The complexity is given to obtain a good result for the different parameters and treatments. The maximum relative error for the real part of $Z(\zeta)$ is 0.005, and that for the imaginary part is 0.02. The other method that is compared is that of Pécseli, where he makes an approximation by sections. It uses four ranges to evaluate the function $Z(\zeta)$ with four different functions. The maximum relative error for the real part of $Z(\zeta)$ is 0.002, and that for its imaginary part is 0.007. The maximum relative error of the real part of $Z'(\zeta)$ is 0.32, and that for the imaginary part is 0.46. The imaginary part of $Z(\zeta)$ and $Z'(\zeta)$ is important in calculating the damping or growth of plasma waves. To improve the approximation, a better fit to the curve is needed. In the proposed method, the power series and the asymptotic expansion are used to find the rational functions that could be used to construct the function $\tilde{Z}(\zeta)$. A better result is obtained when considering polynomials of higher degrees, so an approximation with a polynomial of degree 10 is used. The complication is given by the condition that is required to better adjust the parameters, where the parameter λ is used and as a result the value of $\lambda_0 = 0.006\,517$. The value of the parameter λ is only for the parameters that are developed in this approximation; if we perform another approximation with other parameters of p_i and q_j, another value of lambda will be obtained, which will fit the curve of the function Z. With this parameter, it was possible to find the values of p_i and q_j for the function $\tilde{Z}(\zeta)$ and its derivative that is $\tilde{Z}'(\zeta)$. The values are obtained and represented as shown in Figs. 1 and 5 with Tables I and II. For the function $\tilde{Z}(\zeta)$, it is plotted for $y = 0$, comparing with the other two mentioned methods. The relative error of the real part of the function $\tilde{Z}(\zeta)$ is 0.0035, and its relative error for the imaginary part is 0.0011. Next, we have the graphs of the function $\tilde{Z}'(\zeta)$ compared to the other two methods. The relative error of the real part of the function $\tilde{Z}'(\zeta)$ is 0.026, and its relative error of the imaginary part is 0.000 018. Results are obtained in the range $0 \le x \le 6$ since this range is the objective of the study.

V. CONCLUSION

A new approximation for the plasma dispersion function $Z(\zeta)$ has been found with much better results than those previously published. To determine the new approximation, power series and asymptotic expansions are used simultaneously, and a bridge function between both expansions is determined using rational exponential functions. This procedure is like an extension of complex functions of the so-called multipoint quasi-rational approximation (MPQA) technique. In the present analysis, both complex functions $\tilde{Z}(\zeta)$ and $\tilde{Z}'(\zeta)$ are considered mainly for real values of ζ, where $\zeta = x + iy$ with $y = 0$. Here, the maximum relative error for the real part $\tilde{Z}(\zeta)$ is 0.0035, and that for the imaginary part is 0.0011. In the case of $\tilde{Z}'(\zeta)$, the maximum relative errors are 0.026 and 0.000 018 for the real part of $\tilde{Z}'(\zeta)$ and the imaginary part of $\tilde{Z}'(\zeta)$, respectively. In the present work, the approximation is only one for all values of ζ, which is an important difference with the Pécseli approximation, which is defined using several approximations, each valid only in intervals of the variable ζ. This method (MPQA) was tested for two, four, six, and eight degrees, but the results were not as good as desired for this type of approach. Therefore, it was decided to perform a ten degree approximation to improve the fits to the curve. In this study, no approximations greater than ten degrees were made, so they would remain open problems for future research. The current approximation is better than the approximations of Martin and Pécseli and can be applied with confidence in most applications of the function $Z(\zeta)$ in plasma physics.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

E. Morales-Campaña: Investigation (equal). **P. Martin**: Investigation (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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